Lecture 8

POISSON’S & LAPLACE’S EQUATION

- Two French mathematical physicist Denis Poisson and Pierre Simon de Laplace have derived two fundamental governing differential equations for electrostatics in any medium.

- These equations are very useful mathematical relations for the calculation of electric fields and potentials that can not be computed by Coulomb’s and Gauss’s law.

POISSON'S & LAPLACE'S EQUATION

Consider a continuous distribution of charge in a volume with a charge density ρ. Then total charge

$$Q = \iiint_v \rho \, dv$$

The electric flux linked with the surface enclosing this volume by Gauss's theorem is

$$\phi = \iint_s E dA = \frac{1}{\varepsilon_0} \iiint_v \rho \, dv \qquad ...1$$

But, according to **Gauss's divergence theorem**, the volume integral of divergence of electric field E over a volume V is equal to the surface integral of that field E over the surface S which encloses the given volume i.e.

$$\iint_v \mathrm{div}\vec{\mathrm{E}} \, dv = \iint_s \vec{E} dA \qquad ...2$$

From eq s 1 and 2

$$\iint_v \text{div}\vec{\text{E}}\, dv = \frac{1}{\varepsilon_0}\iiint_v \rho\, dv \qquad ...3$$

$$\iiint_v \left\{\text{div}\vec{\text{E}} - \frac{\rho}{\varepsilon_0}\right\} = 0$$

$$\text{div}\vec{\text{E}} - \frac{\rho}{\varepsilon_0} = 0$$

$$\text{div}\vec{\text{E}} = \frac{\rho}{\varepsilon_0} \qquad \text{Poisson's equation} \qquad ...4$$

Poisson’s equation can also be expressed in terms of electric potential V.
We know that the intensity of electric field is the negative gradient of electric potential i.e.

$$\vec{\text{E}} = -gradV = -\nabla V$$

$$\therefore \qquad \text{div}\vec{\text{E}} = \nabla \bullet E = \nabla \bullet \left(-\nabla V\right) = -\nabla^2 V \quad ...5$$

From eq s 4 and 5

$$\nabla^2 V = -\frac{\rho}{\varepsilon_0} \qquad \ldots 6$$

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = -\frac{\rho}{\varepsilon_0}$$

This is the other form of Poisson's equation.

If we consider a ***charge free*** region then ρ = 0
Then eq 4

$$div\,\vec{E} = 0$$

$$\nabla^2 V = 0$$

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0$$

this is called the Laplace's equation

Cartesian coordinate system

$$\nabla^2 V = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0$$

Cylindrical Coordinate System

$$\nabla^2 V = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial V}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 V}{\partial \phi^2} + \frac{\partial^2 V}{\partial z^2} = 0$$

Spherical Coordinate System

$$\nabla^2 V = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial V}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial V}{\partial \theta}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial^2 V}{\partial \phi^2} = 0$$

Example:

Consider a parallel conductor where V = 0 at z = 0 and V = 100 Volts at z = d. Calculate potential as a Function of z.

$$\frac{\partial^2 V}{\partial x^2}+\frac{\partial^2 V}{\partial y^2}+\frac{\partial^2 V}{\partial z^2}=0$$

Since V is not the function of x and y so Laplace's equation reduces to

$$\frac{\partial^2 V}{\partial z^2}=0$$

V = A z +B

Using given Conditions V= 0 at z = 0 Provide B = 0

V =100 at z =d gives A = 100/d

V = 100(z/d) Volts