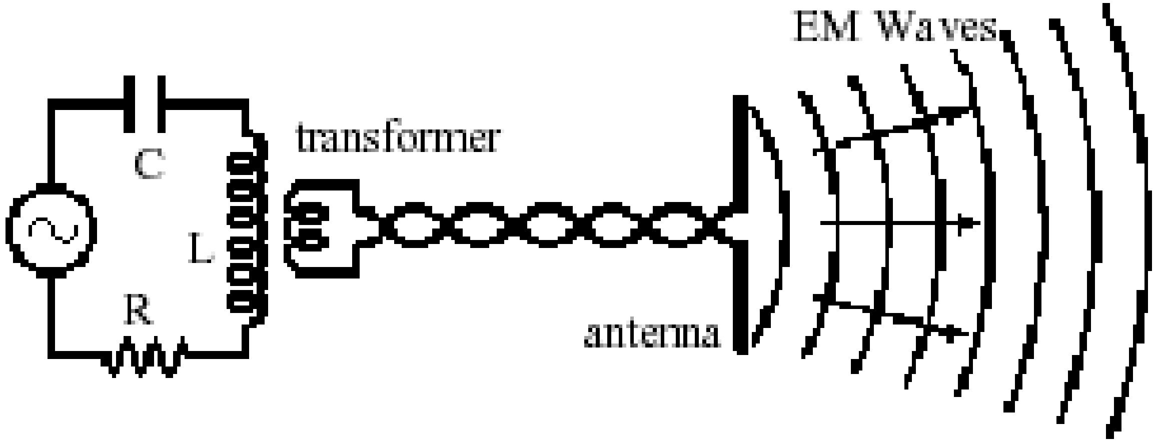
Lecture 12

Poynting vector and Poynting theorem

The production and propagation of electromagnetic waves

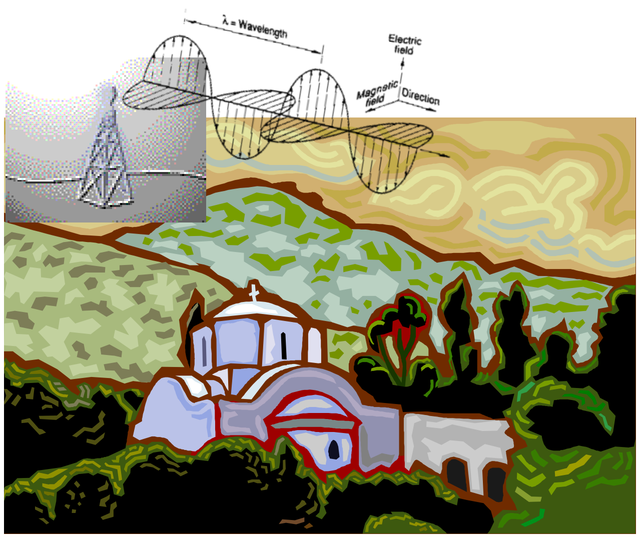
- Let's put them both together: *we obtain changing electric and magnetic fields that continuously produce each other!*



The oscillating charges in the antenna set up electric and magnetic fields.

Power in a wave

- A wave **carries power** and **transmits** it's power wherever it goes



• The rate of flow of energy per unit area in wave is given by the *Poynting vector.*

•Symbolized by vector S

$$\vec{S} = \vec{E} \times \vec{H}$$

Poynting Vector Derivation

Taking scalar product of Maxwell 's eq 3 with H and eq 4 with E

$$H\cdot\nabla\times E=-H\cdot\frac{\partial B}{\partial t}\qquad ...1$$

$$\mathrm{E}\cdot\left(\nabla\times\mathrm{H}\right)=\mathrm{E}\cdot\mathrm{J}+\mathrm{E}\cdot\frac{\partial D}{\partial t}\qquad ...2$$

Subtracting eq 2 from eq 1

$$H\cdot\nabla\times E-\mathrm{E}\cdot\left(\nabla\times\mathrm{H}\right)=-H\cdot\frac{\partial B}{\partial t}-\ \mathrm{E}\cdot\mathrm{J}-\mathrm{E}\cdot\frac{\partial D}{\partial t}$$

$$H\cdot\nabla\times E-\mathrm{E}\cdot\left(\nabla\times\mathrm{H}\right)=-\left[H\cdot\frac{\partial B}{\partial t}+\mathrm{E}\cdot\frac{\partial D}{\partial t}\right]-\ \mathrm{E}\cdot\mathrm{J}\qquad ...3$$

Now using vector identity

$$\nabla\cdot\left(A\times B\right)=B\cdot\left(\nabla\times A\right)-A\cdot\left(\nabla\times B\right)\text{or in this case}:$$

$$\nabla\cdot\left(E\times H\right)=H\cdot\left(\nabla\times E\right)-E\cdot\left(\nabla\times H\right)$$

Eq 3 can be written as

$$\nabla\cdot(E\times \mathrm{H}) = -\left[H\cdot\frac{\partial B}{\partial t} + \mathrm{E}\cdot\frac{\partial D}{\partial t}\right] - \mathrm{E}\cdot \mathrm{J} \qquad ...4$$

Using relation B=μH and D=εE, in eq 4

$$\nabla\cdot(E\times \mathrm{H}) = -\left[H\cdot\frac{\partial(\mu H)}{\partial t} + \mathrm{E}\cdot\frac{\partial(\varepsilon E)}{\partial t}\right] - \mathrm{E}\cdot \mathrm{J}$$

$$= -\mu H\cdot\frac{\partial H}{\partial t} - \varepsilon\mathrm{E}\cdot\frac{\partial E}{\partial t} - \mathrm{E}\cdot \mathrm{J} \qquad ....5$$

But

$$H\cdot\frac{\partial H}{\partial t} = \frac{1}{2}\frac{\partial(H)^2}{\partial t} \quad \text{and} \quad E\cdot\frac{\partial E}{\partial t} = \frac{1}{2}\frac{\partial(E)^2}{\partial t}$$

$$\nabla\cdot(E\times \mathrm{H}) = -\frac{\mu}{2}\frac{\partial(H)^2}{\partial t} - \frac{\varepsilon}{2}\frac{\partial(E)^2}{\partial t} - \mathrm{E}\cdot \mathrm{J}$$

$$\nabla\cdot\left(E\times \mathrm{H}\right)=\frac{\partial}{\partial t}\left[\frac{\mu H^{2}}{2}+\frac{\varepsilon E^{2}}{2}\right]-\ \ \mathrm{E}\cdot\mathrm{J}\quad \ldots 6$$

Now integrating eq 6 over a volume V bounded by surface S

$$\int_{v}\nabla\cdot\left(E\times H\right)dv=-\frac{\partial}{\partial t}\int_{v}\left(\frac{\varepsilon}{2}E^{2}+\frac{\mu}{2}H^{2}\right)dv-\int_{v}E\cdot J\ dv\quad \ldots 7$$

Using Gauss Divergence theorem $\int_{v}\nabla\cdot\left(E\times H\right)dv=\oint_{s}\left(E\times H\right)ds$

Eq 7 becomes

$$\oint_{S}\left(E\times H\right)\cdot dS=-\frac{\partial}{\partial t}\int_{v}\left(\frac{\varepsilon}{2}E^{2}+\frac{\mu}{2}H^{2}\right)dv-\int_{v}E\cdot J dv\quad \ldots 9$$

Which means that the total power coming out of a volume is either due to the electric or magnetic field energy variations or is lost in ohmic losses.

Poynting Vector

- Waves carry energy and information

- Poynting says that the net power flowing out of a given volume is = to the decrease in time in energy stored minus the conduction losses.

$$\vec{S} = \vec{E} \times \vec{H} \quad [\mathrm{W/m^2}]$$

- The Poynting vector has the same direction as the direction of propagation.

Poynting Vector for an em wave propagating in free space

$$\vec{S} = \vec{E} \times \vec{H} = \vec{E} \times \frac{\hat{n} \times \vec{E}}{\mu_0 c} = \frac{1}{\mu_0 c} \vec{E} \times \hat{n} \times \vec{E}$$

$$= \frac{1}{\mu_0 c} \left(\vec{E} \cdot \vec{E} \right) \hat{n} - \left(\vec{E} \cdot \hat{n} \right) \vec{E}$$

$$= \frac{1}{\mu_0 c} \vec{E}^2 \hat{n} \qquad \because \left(\vec{E} \cdot \hat{n} = 0, E \text{ being } \perp \text{ to } \hat{\text{n}} \right)$$

$$= \frac{\vec{E}^2}{Z_0} \hat{n}$$

For a plane em wave of angular frequency , ω the average value of S over a complete cycle is given by

$$\langle S \rangle = \frac{\langle \vec{E}^2 \rangle}{Z_0} \hat{n} = \frac{\left\langle E_0 e^{(i\vec{k}.\vec{r} - i\omega t)^2} \right\rangle_{real}}{Z_0} \hat{n}$$

$$= \frac{1}{Z_0} E_0{}^2 \left\langle \cos^2(\omega t - \vec{k}.\vec{r}) \right\rangle \hat{n}$$

Average is obtained by integrating over a time period and dividing by T

$$= \frac{1}{Z_0} \frac{E_0{}^2}{2} \hat{n} \qquad \left\langle \cos^2(\omega t - \vec{k}.\vec{r}) \right\rangle = 1/2$$

$$= \frac{1}{Z_0} E_{rms} \hat{n} \qquad E_{rms} = \frac{E_0}{\sqrt{2}}$$

The direction of Poynting vector is along the direction of propagation of em wave. This means that the flow of energy in a plane em wave in free space is along the direction of wave

The Poynting Vector

Energy transport is defined by the Poynting vector S as:

$$\vec{S} \equiv \frac{\vec{E} \times \vec{B}}{\mu_0}$$

The direction of S is the direction of propagation of the wave

$$S = \frac{EB}{\mu_0} = \frac{E^2}{\mu_0 c} = \frac{E^2}{Z_0} = \frac{E^2}{377\Omega}$$

Electrostatic Boundary Conditions

- **At any point on the boundary,**
  - the components of $\boldsymbol{E_1}$ and $\boldsymbol{E_2}$ tangential to the boundary are equal

$$E_{1t} = E_{2t}$$

  - the components of $\boldsymbol{D_1}$ and $\boldsymbol{D_2}$ normal to the boundary are equal

$$D_{1t} = D_{2t}$$

Hayt p-143

Magnetic Boundary Conditions

- The normal component of $\boldsymbol{B_1}$ and $\boldsymbol{B_2}$ continuous across a interface:

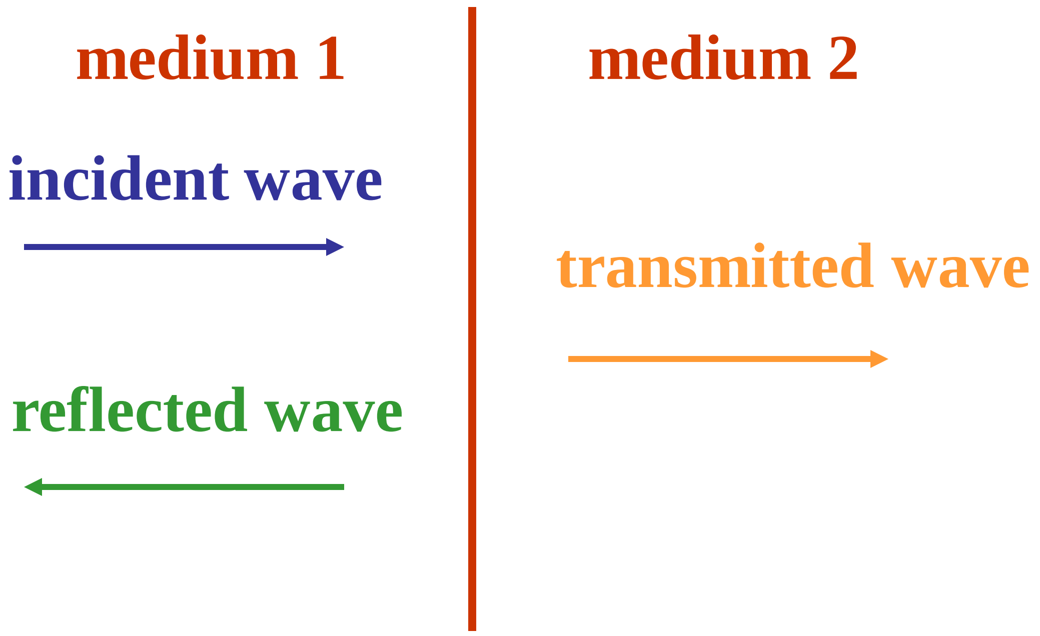
$$B_{1n} = B_{2n}$$

- The tangential component of a $\boldsymbol{H_1}$ and $\boldsymbol{H_2}$ to the boundary are equal

$$H_{1t} = H_{2t}$$

Hayt p-281

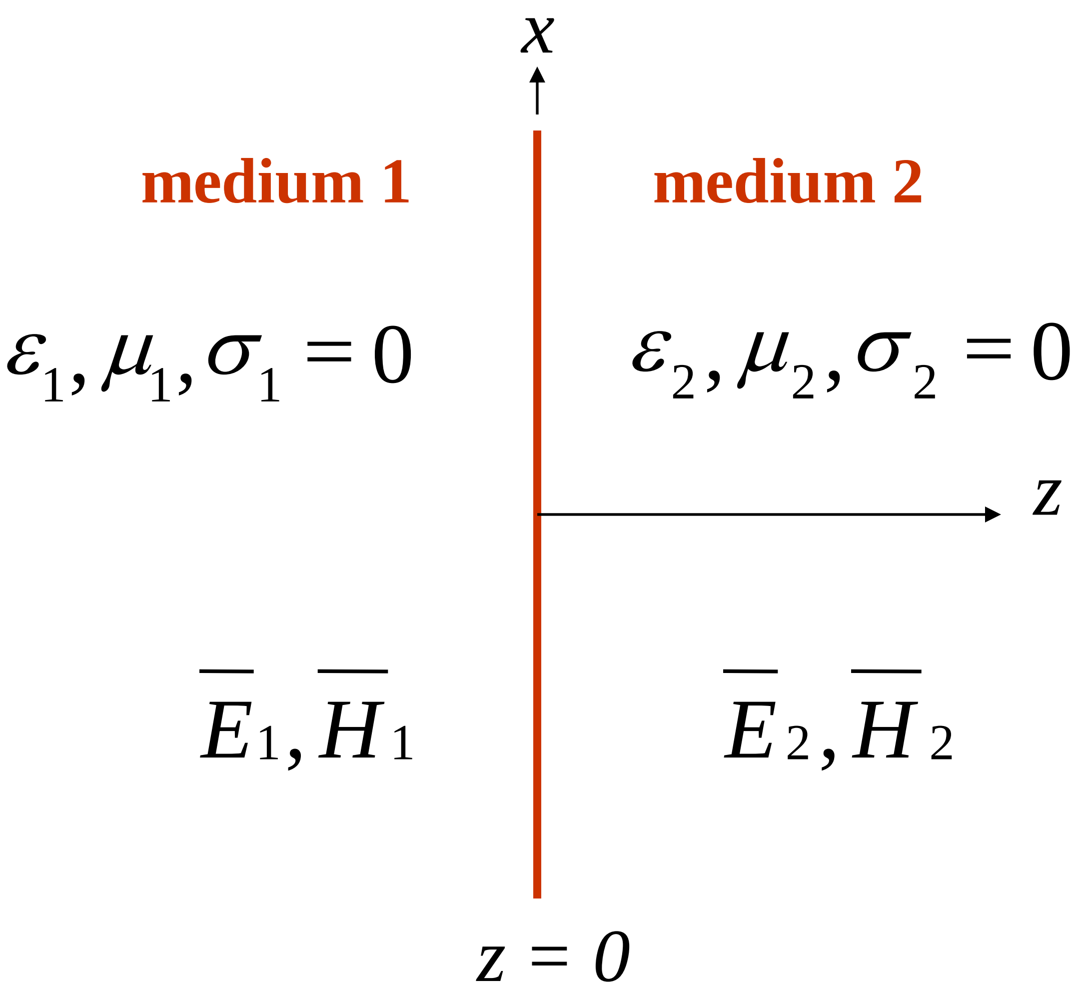
Reflection and Transmission of Waves at Planar Interfaces



Normal Incidence

- Consider both medium 1 and medium 2
- Let us place the boundary between the two media in the $z = 0$ plane, and consider an incident plane wave which is traveling in the $+z$-direction.
- we assume that the electric field of the incident wave is in the x-direction.

Normal Incidence



Normal Incidence

- Incident wave

known

$$\overline{E}_i = \hat{a}_x E_{i0} e^{-j\beta_1 z}$$

$$\overline{H}_i = \frac{1}{\eta_1} \hat{a}_z \times \overline{E}_i = \hat{a}_y \frac{E_{i0}}{\eta_1} e^{-j\beta_1 z}$$

$$\beta_1 = \omega\sqrt{\varepsilon_1 \mu_1} \qquad \eta_1 = \sqrt{\frac{\mu_1}{\varepsilon_1}}$$

Normal Incidence

- Reflected wave

unknown

$$\overline{E}_r = \hat{a}_x E_{r0} e^{+j\beta_1 z}$$

$$\overline{H}_r = \frac{1}{\eta_1}\left(-\hat{a}_z\right)\times \overline{E}_r = -\hat{a}_y \frac{E_{r0}}{\eta_1} e^{+j\beta_1 z}$$

Normal Incidence

- Transmitted wave

unknown

$$\overline{E}_t = \hat{a}_x E_{t0} e^{-j\beta_2 z}$$

$$\overline{H}_t = \frac{1}{\eta_2} \hat{a}_z \times \overline{E}_t = \hat{a}_y \frac{E_{t0}}{\eta_2} e^{-j\beta_2 z}$$

$$\beta_2 = \omega\sqrt{\varepsilon_2 \mu_2} \qquad \eta_2 = \sqrt{\frac{\mu_2}{\varepsilon_2}}$$

Normal Incidence

- The total electric and magnetic fields in medium 1 are

$$\overline{E}_1 = \overline{E}_i + \overline{E}_r = \hat{a}_x \left[E_{i0} e^{-j\beta_1 z} + E_{r0} e^{+j\beta_1 z} \right]$$

$$\overline{H}_1 = \overline{H}_i + \overline{H}_r = \hat{a}_y \left[\frac{E_{i0}}{\eta_1} e^{-j\beta_1 z} - \frac{E_{r0}}{\eta_1} e^{+j\beta_1 z} \right]$$

Normal Incidence

- The total electric and magnetic fields in medium 2 are

$$\overline{E}_2 = \overline{E}_t = \hat{a}_x E_{t0} e^{-j\beta_2 z}$$

$$\overline{H}_2 = \overline{H}_t = \hat{a}_y \frac{E_{t0}}{\eta_2} e^{-j\beta_2 z}$$

Normal Incidence

- To determine the unknowns E_{r0} and E_{t0}, we must enforce the BCs at $z = 0$:

$$\overline{E}_1(z=0) = \overline{E}_2(z=0)$$

$$\overline{H}_1(z=0) = \overline{H}_2(z=0)$$

Normal Incidence)

- From the BCs we have

$$E_{i0} + E_{r0} = E_{t0}$$

or

$$\frac{E_{i0}}{\eta_1} - \frac{E_{r0}}{\eta_1} = \frac{E_{t0}}{\eta_2}$$

$$E_{r0} = \frac{\eta_2 - \eta_1}{\eta_2 + \eta_1} E_{i0}, \qquad E_{t0} = \frac{2\eta_2}{\eta_2 + \eta_1} E_{i0}$$

Reflection and Transmission Coefficients

- Define the ***reflection coefficient*** as

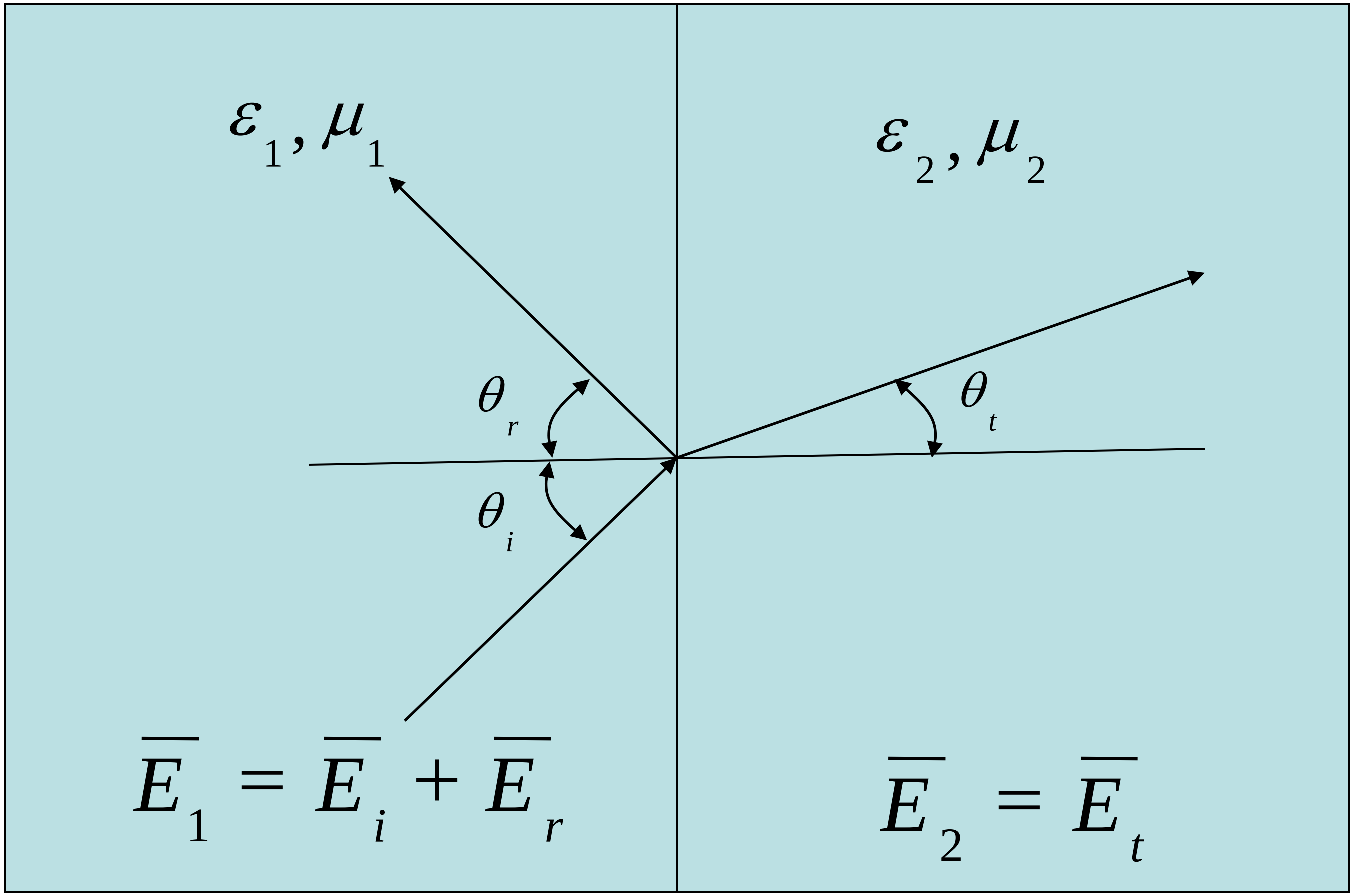
$$\Gamma = \frac{E_{r0}}{E_{i0}} = \frac{\eta_2 - \eta_1}{\eta_2 + \eta_1}$$

- Define the ***transmission coefficient*** as

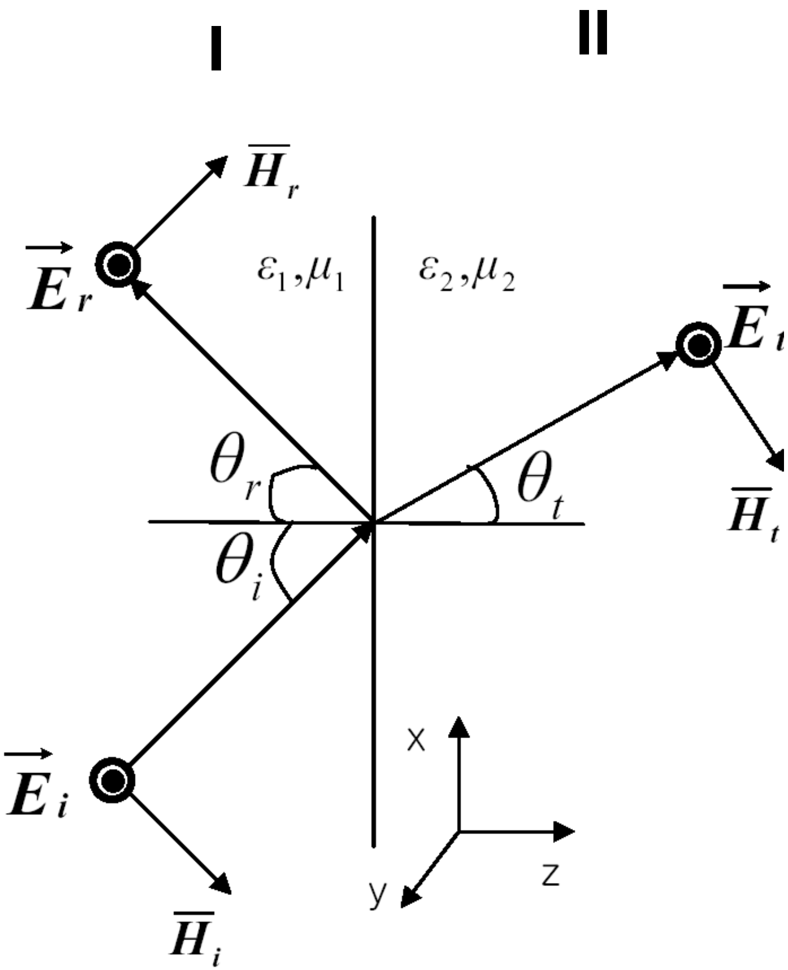
$$\tau = \frac{E_{t0}}{E_{i0}} = \frac{2\eta_2}{\eta_2 + \eta_1}$$

Oblique Incidence

$$z = 0$$



Perpendicular Polarization



$$\overline{E}_i = E_{i0}\exp(-j\beta_1 z)\, a_x$$

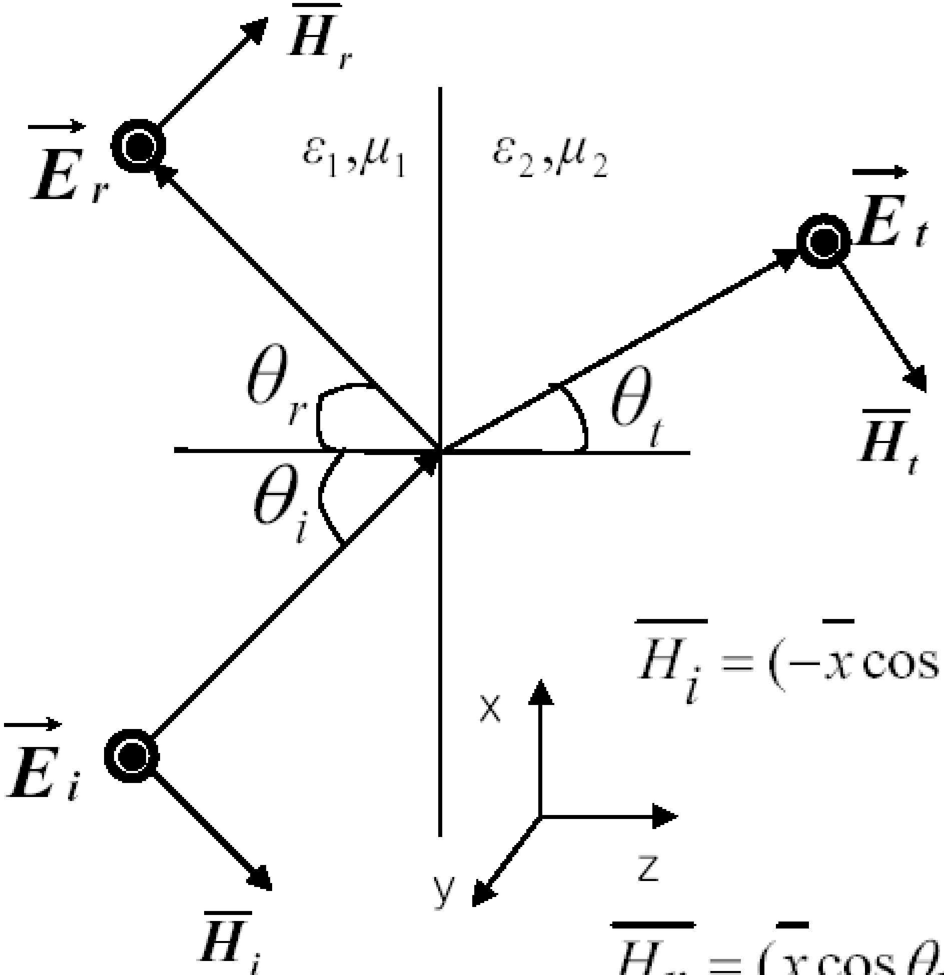
$$\overline{E_i} = \overline{y} E_i \exp(-j\beta_{1x} x)\exp(-j\beta_{1z} z)$$

$$\overline{E_r} = \overline{y}\,\Gamma E_i \exp(-j\beta_{rx} x)\exp(j\beta_{rz} z)$$

$$\overline{E_t} = \overline{y}\,\tau E_i \exp(-j\beta_{2x} x)\exp(-j\beta_{2z} z)$$

$$\Gamma = \frac{E_r}{E_i}$$

$$\tau = \frac{E_t}{E_i}$$



$$\overline{H_i} = (-\bar{x}\cos\theta_i + \bar{z}\sin\theta_i)\frac{E_i}{\eta_1}\exp(-j\beta_{1x}x)\exp(-j\beta_{1z}z)$$

$$\overline{H_r} = (\bar{x}\cos\theta_r + \bar{z}\sin\theta_r)\frac{\Gamma E_i}{\eta_1}\exp(-j\beta_{rx}x)\exp(j\beta_{rz}z)$$

$$\overline{H_t} = (-\bar{x}\cos\theta_t + \bar{z}\sin\theta_t)\frac{\tau E_i}{\eta_2}\exp(-j\beta_{2x}x)\exp(-j\beta_{2z}z)$$

B.C.'s at $z=0$

1) $\bar{E}_{tan}$ coutinuous $(\overline{E_i}+\overline{E_r}=\overline{E_t})$

$$\overline{E_i} = \bar{y} E_i \exp(-j\beta_{1x}x)\exp(-j\beta_{1z}z)$$

$$\overline{E_r} = \bar{y}\, \Gamma E_i \exp(-j\beta_{rx}x)\exp(j\beta_{rz}z)$$

$$\overline{E_t} = \bar{y}\, \tau E_i \exp(-j\beta_{2x}x)\exp(-j\beta_{2z}z)$$

$$\exp(-j\beta_{1x}x) + \Gamma\exp(-j\beta_{rx}x) = \tau\exp(-j\beta_{2x}x) \quad \text{......(A)}$$

$$\beta_{1x} = \beta_{rx} = \beta_{2x}$$

$$1+\Gamma=\tau \quad \text{.....(B)}$$

B.C.'s at $z=0$

2) $\bar{H}_{tan}$ continuous $(\bar{H}_i + \bar{H}_r = \bar{H}_t)$

$$-\frac{\cos\theta_i}{\eta_1} + \frac{\cos\theta_i}{\eta_1}\Gamma = -\frac{\cos\theta_t}{\eta_2}\tau$$ **……(C)**

With $1+\Gamma=\tau$,

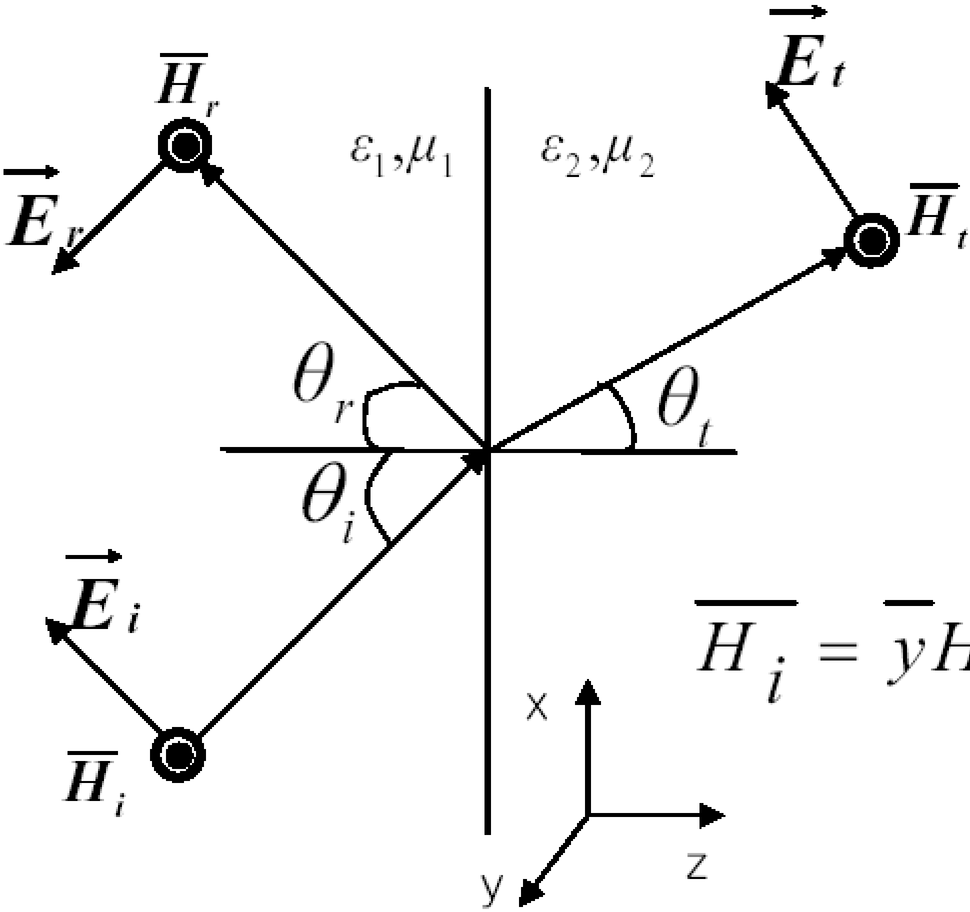
reflection coefficient

$$\Gamma_\perp = \frac{\eta_2/\cos\theta_t - \eta_1/\cos\theta_i}{\eta_2/\cos\theta_t + \eta_1/\cos\theta_i}$$

transmission coefficient

$$\tau_\perp = \frac{2\eta_2/\cos\theta_t}{\eta_2/\cos\theta_t + \eta_1/\cos\theta_i}$$

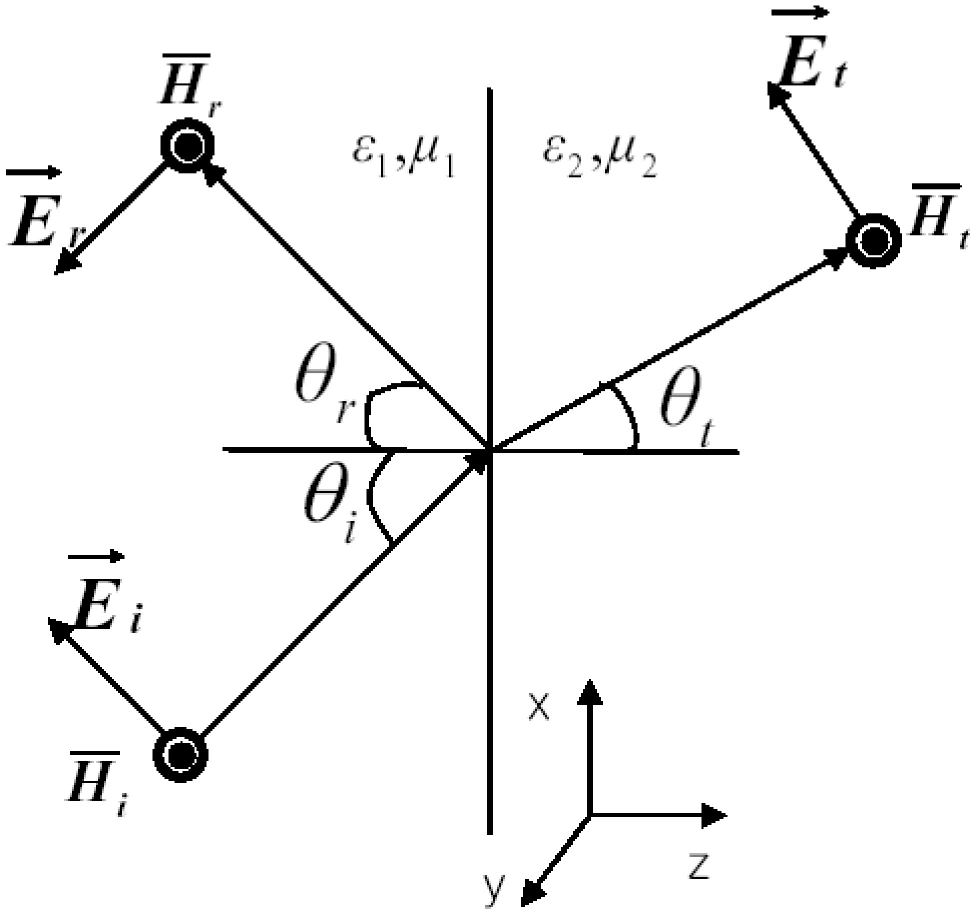
Parallel Polarization



$$\overline{H_i} = \overline{y} H_i \exp(-j\beta_{1x} x) \exp(-j\beta_{1z} z)$$

$$\overline{H_r} = \overline{y} H_r \exp(-j\beta_{rx} x) \exp(j\beta_{rz} z)$$

$$\overline{H_t} = \overline{y} H_t \exp(-j\beta_{2x} x) \exp(-j\beta_{2z} z)$$



$$E = H\eta$$

$$\overline{E_i} = (\overline{x}\cos\theta_i - \overline{z}\sin\theta_i)H_i\eta_1 \exp(-j\beta_{1x}x)\exp(-j\beta_{1z}z)$$

$$\overline{E_r} = (-\overline{x}\cos\theta_r - \overline{z}\sin\theta_r)H_r\eta_1 \exp(-j\beta_{rx}x)\exp(j\beta_{rz}z)$$

$$\overline{E_t} = (\overline{x}\cos\theta_t - \overline{z}\sin\theta_t)H_t\eta_2 \exp(-j\beta_{2x}x)\exp(-j\beta_{2z}z)$$

$$\frac{H_r}{H_i} = -\Gamma \qquad \frac{H_t}{H_i} = \frac{E_t/\eta_2}{E_i/\eta_1} = \tau\,\frac{\eta_1}{\eta_2}$$

B.C.'s at z=0

1) $\bar{H}_{\tan}$ coutinuous $(\bar{H}_i + \bar{H}_r = \bar{H}_t)$

$$\overline{H_i} = \bar{y} H_i \exp(-j\beta_{1x}x)\exp(-j\beta_{1z}z)$$

$$\overline{H_r} = \bar{y} H_r \exp(-j\beta_{rx}x)\exp(j\beta_{rz}z)$$

$$\overline{H_t} = \bar{y} H_t \exp(-j\beta_{2x}x)\exp(-j\beta_{2z}z)$$

$$\exp(-j\beta_{1x}x) - \Gamma\exp(-j\beta_{rx}x) = \frac{\eta_1}{\eta_2}\tau\exp(-j\beta_{2x}x)$$

$$\beta_{1x} = \beta_{rx} = \beta_{2x}$$

$$1 - \Gamma = \frac{\eta_1}{\eta_2}\tau \qquad \text{.....(A)}$$

B.C.'s at $z = 0$

2) $\bar{E}_{\tan}$ continuous

$$\cos\theta_i + \Gamma\cos\theta_i = \tau\cos\theta_t \qquad \text{.........(B)}$$

reflection coefficient

$$\Gamma_{\parallel} = \frac{\eta_2 \cos\theta_t - \eta_1 \cos\theta_i}{\eta_2 \cos\theta_t + \eta_1 \cos\theta_i}$$

transmission coefficient

$$\tau_{\parallel} = \frac{2\eta_2 \cos\theta_i}{\eta_2 \cos\theta_t + \eta_1 \cos\theta_i}$$