Lecture 6

We will use a "charge density" to describe the distribution of charge.
This charge density will be different depending on the geometry

- If a charge Q is uniformly distributed throughout a volume V, the **volume charge density** ρ is defined by $\rho = \frac{Q}{V}$
  where ρ has units of coulombs per cubic meter (C/m^3).

- If a charge Q is uniformly distributed on a surface of area A, the **surface charge density** σ (lowercase Greek sigma) is defined by $\sigma = \frac{Q}{A}$
  where σ has units of coulombs per square meter (C/m^2).

- If a charge Q is uniformly distributed along a line of length ℓ, the **linear charge density** λ is defined by $\lambda = \frac{Q}{\ell}$
  where λ has units of coulombs per meter (C/m).

Symbol	Name	Unit
λ	Charge per length	C/m
σ	Charge per area	C/m^2
ρ	Charge per volume	C/m^3

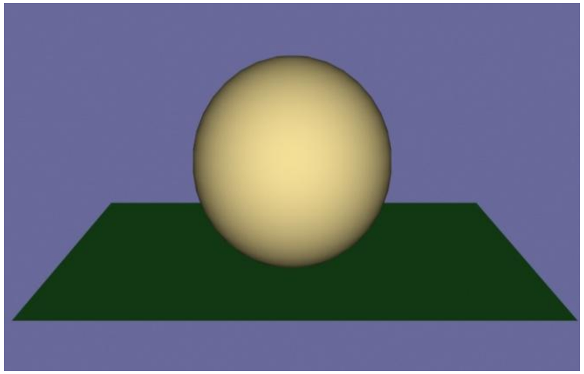
Definitions

- Symmetry—The balanced structure of an object, the halves of which are alike

Open and Closed Surfaces

A rectangle is an open surface — it does NOT contain a volume

A sphere is a closed surface — it DOES contain a volume



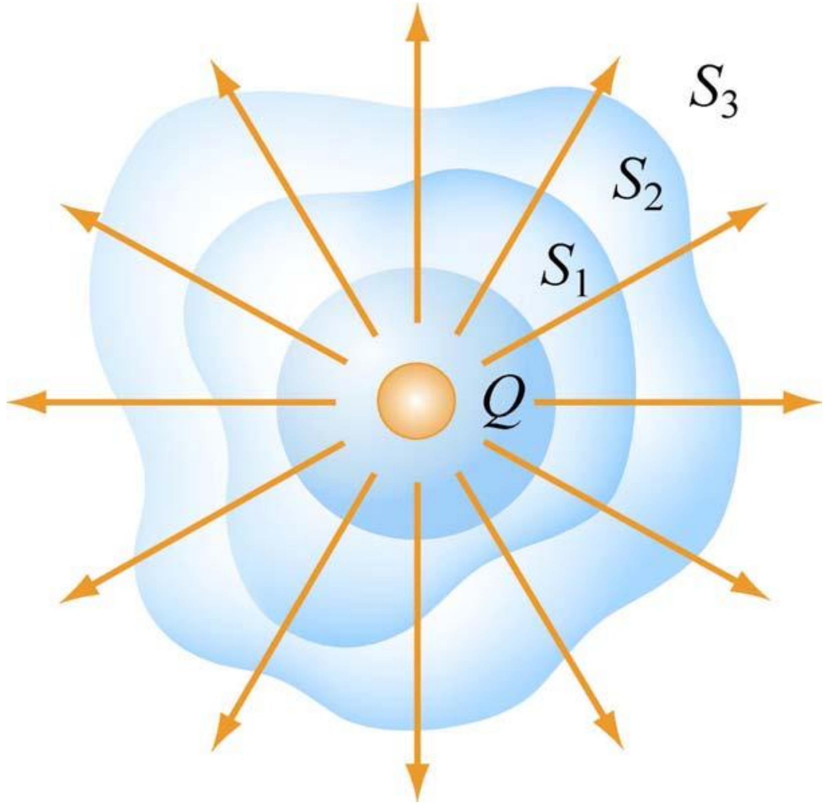
- Closed surface—A surface that divides space into an inside and outside region, so one can't move from one region to another without crossing the surface
- Gaussian surface—A hypothetical closed surface that has the same symmetry as the problem we are working on—note this is not a real surface it is just an mathematical one

What is Gauss's Law?

Gauss's Law does not tell us anything new, it is NOT a new law of physics, but another way of expressing Coulomb's Law

Gauss's Law is sometimes easier to use than Coulomb's Law, especially if there is lots of symmetry in the problem

Gauss's Law – The Idea



The total "flux" of field lines penetrating any of these surfaces is the same and depends only on the amount of charge inside

Gauss's Law – The Equation

According to the Gauss's law, total number of electric line of force passing normally through a closed surface of any shape in an electric field (i.e. the total electric flux) is equal to $1/\varepsilon_0$ times the total charge present with in that surface.

$$\Phi_E = \underset{\substack{\text{closed} \\ \text{surfaceS}}}{\oiint} \vec{\mathbf{E}} \cdot d\vec{\mathbf{A}} = \frac{q_{in}}{\varepsilon_0}$$

Electric flux ϕ_E is proportional to charge inside the volume enclosed by S

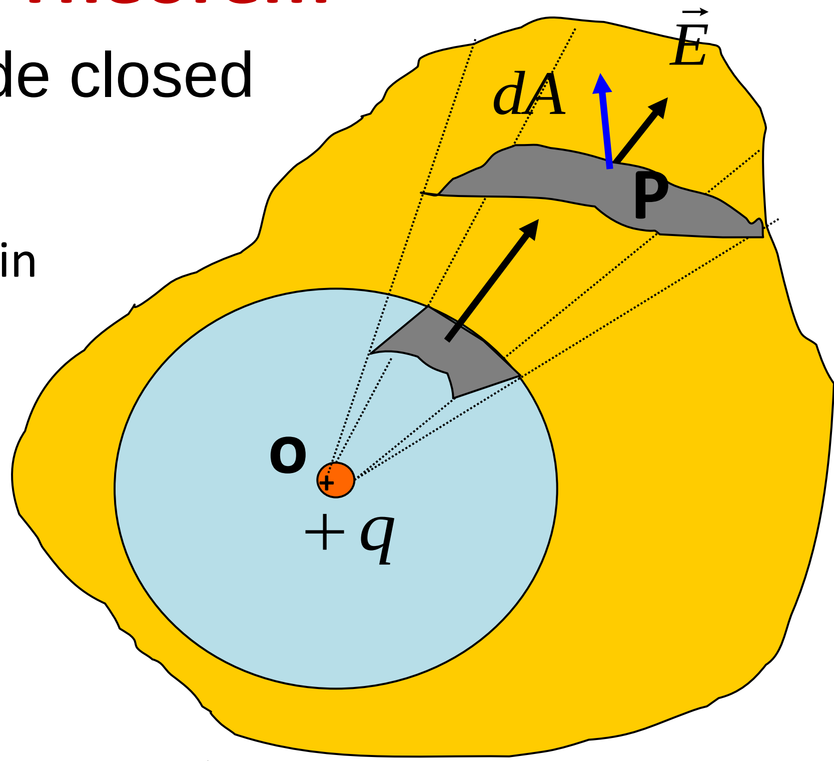
If there is no charge within the surface or charge outside the surface then

$$\Phi_E = \oiint \vec{E} \cdot d\vec{A} = 0$$

Note: Integral must be over closed surface

Proof of Gauss's Theorem

❑ Case 1: Single positive charge inside closed surface



• Let a point charge q is situated at a point O within the closed surface S.

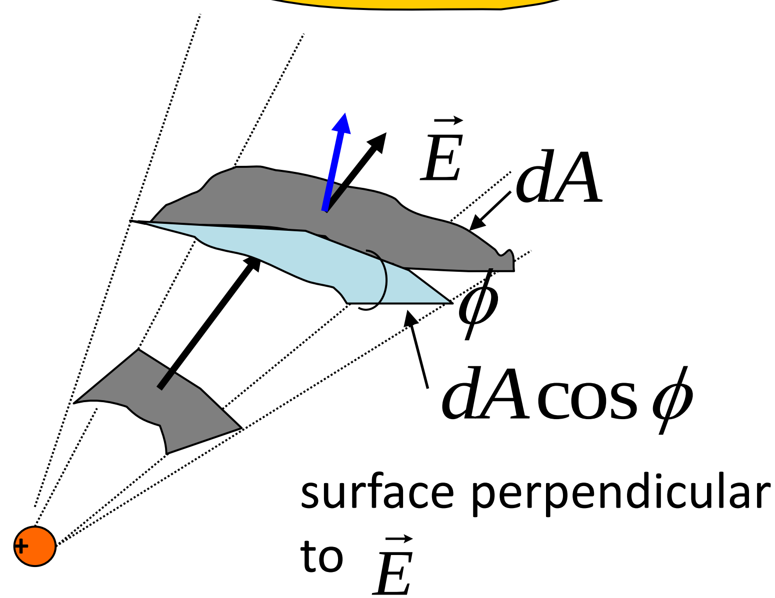
• Consider a small area dA around a point P on the surface. The area dA is directed normal to the surface.

•The distance of the point P from the point O is r

• Then the electric flux passing through surface area dA

$$E = \frac{1}{4\pi\varepsilon_0}\frac{q}{r^2} \qquad \text{....1}$$

If ϕ is the angle between the electric field E and area dA then



Then the electric flux passing through surface area dA whose normal makes an angle ϕ with the field

$$d\Phi_E = E \bullet dA$$

$$d\Phi_E = EdA\cos\phi \qquad ...2$$

Substituting the value of E from eq 1

$$d\Phi_E = \frac{q}{4\pi\varepsilon_0 r^2} dA\cos\phi \qquad ....3$$

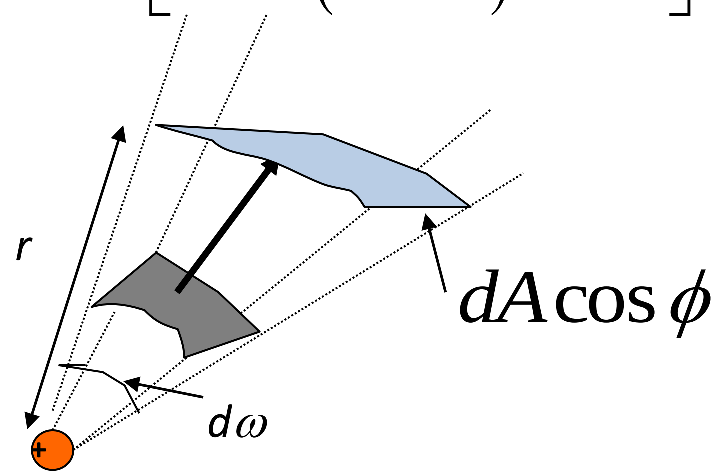
By definition solid angle subtended by a area dA at point O

$$d\omega = \frac{dA\cos\phi}{r^2}$$

$$\left[d\omega = \frac{surface \quad area}{(radius)^2} \right]$$

Therefore eq 3 reduces to

$$d\Phi_E = \frac{q}{4\pi\varepsilon_0} d\omega$$



Hence the electric flux through whole of the closed surface

$$\Phi_E = \sum d\phi = \frac{\sum q\,d\omega}{4\pi\varepsilon_0}$$

(Solid angle subtended by the entire closed surface at an internal point O is 4π)

$$\Phi_E = \frac{q}{4\pi\varepsilon_0}\sum d\omega$$

$$\Phi_E = \frac{q}{4\pi\varepsilon_0}\cdot 4\pi = \frac{q}{\varepsilon_0}$$

$$\Phi_E = \oint \vec{E}\cdot d\vec{A} = \frac{q}{\varepsilon_0}$$

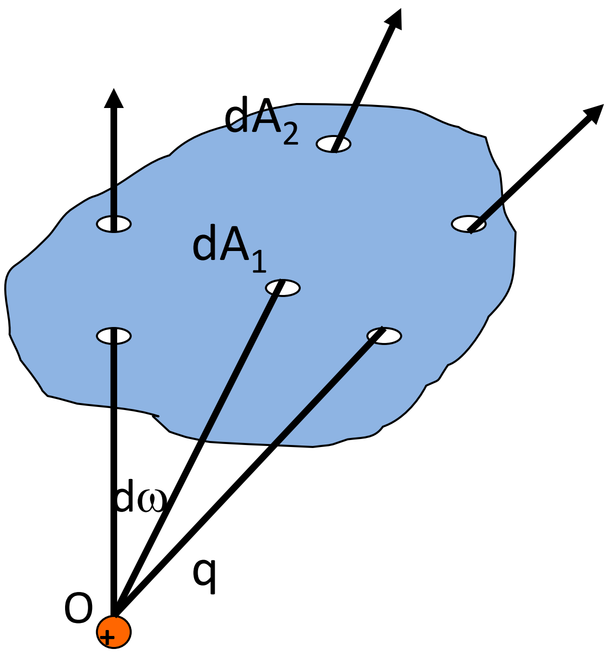
Hence the total electric flux through any closed surface is equal to $1/\varepsilon_0$ times of total charge enclosed within the surface which is Gauss law.

Gauss's Law

- Case 2: Single positive charge outside closed surface

- Let a point charge q is situated at a point O outside the closed surface S.

- Consider a cone subtending a solid angle $d\omega$ at a point O, which intercepts the given surface at dA_1 and dA_2.



Electric flux entering through area dA_1

$$d\Phi_{A_1} = -\frac{q}{4\pi\varepsilon_0} d\omega \quad ...1$$

Electric flux coming out through area dA_2

$$d\Phi_{A_2} = \frac{q}{4\pi\varepsilon_0} d\omega \quad ...2$$

Total electric flux through a closed surface enclosed by the intercepts dA_1 and dA_2

$$d\Phi = d\Phi_{A_1} + d\Phi_{A_2} \quad ...3$$

Electric field lines that go in come out.

Electric field lines can begin or end inside a region of space only when there is charge in that region.

Putting the value of dA_1 and dA_2 from equation (1) and (2) in eq (3)

$$d\Phi = \frac{q}{4\pi\varepsilon_0} d\omega + \left[-\frac{q}{4\pi\varepsilon_0} d\omega \right]$$

$$\Phi_E = 0$$

As there is no charge within the surface, the total electric flux through the whole surface is zero

Applying Gauss's Law

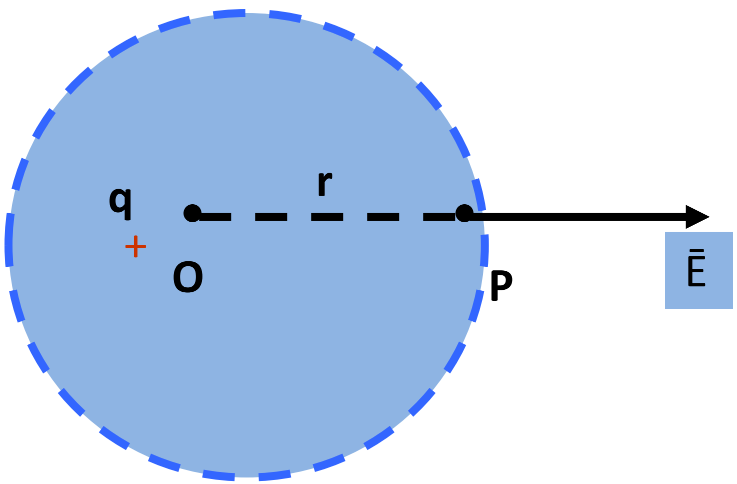
1. Identify regions in which to calculate E field.
2. Choose Gaussian surfaces S: Symmetry
3. Calculate $\Phi_E = \underset{S}{\oiint} \vec{\mathbf{E}} \cdot d\vec{\mathbf{A}}$
4. Calculate q_{in}, charge enclosed by surface S
5. Apply Gauss's Law to calculate E:

$$\boxed{\Phi_E = \underset{\text{closed surfaceS}}{\oiint} \vec{\mathbf{E}} \cdot d\vec{\mathbf{A}} = \frac{q_{in}}{\varepsilon_0}}$$

Applications of gauss's theorem

(a) Cases of spherical symmetry

(i) Field due to point charge



Electric flux through the spherical surface

$$\phi_E = \oint_A E \cdot dA = EdA = E(4\pi r^2)$$

Charge enclosed by surface Q_{encl} = q

By Gauss theorem $$\oint_A E \cdot dA = \frac{Q_{encl}}{\varepsilon_0}$$

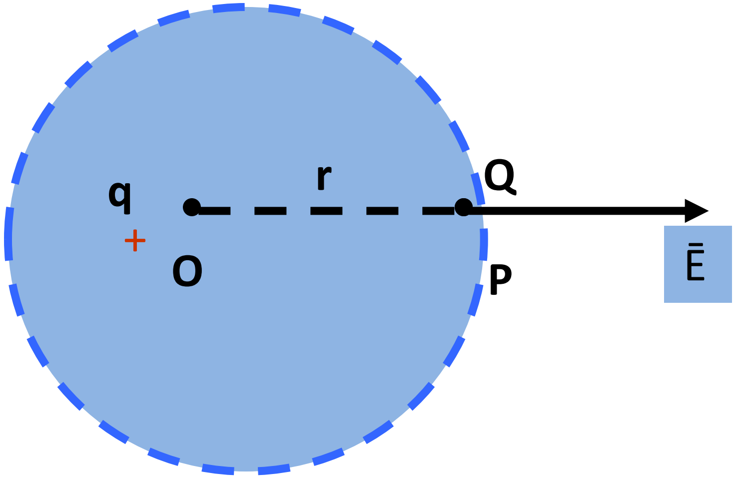
$$E.4\pi r^2 = \frac{1}{\varepsilon_0} \times q$$

$$E = \frac{1}{4\pi r^2 \varepsilon_0} \times q$$

- Electric field E at each point of surface is same & directed outward

Derivation of Coulomb's law from Gauss's theorem

•Now if a point charge Q is kept at the point P on the surface of sphere, the force acting on the charge Q is



$$F = QE = Q\frac{q}{4\pi r^2 \varepsilon_0}$$

$$F = \frac{1}{4\pi\varepsilon_0}\frac{Qq}{r^2}$$

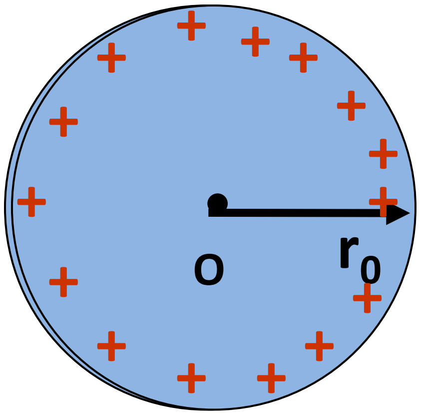
•Thus the force acting between the two point charges Q and q at a separation r is

$$F = \frac{1}{4\pi\varepsilon_0}\frac{Qq}{r^2} \qquad \text{Coulombs law}$$

(ii) Electric Field due to uniformly charged conducting sphere (Spherical shell)

•Let a conducting sphere of radius r_0 is uniformly charged with a charge +q.
•As the material of the sphere is conducting, total charge resides on the outer surface of the sphere and there is no charge with in the sphere.

•We are to find the electric field intensity at a point P, at distance r from the center of the sphere in the following three cases.

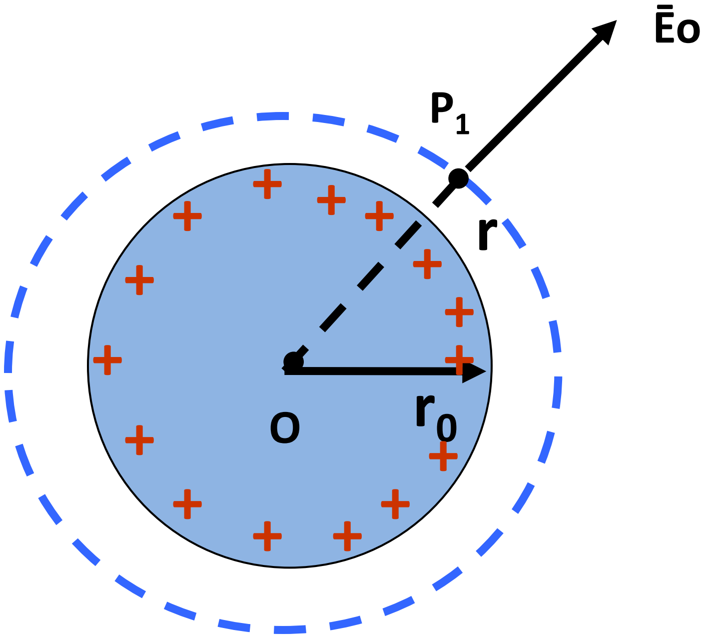


- **When the point P is out side the sphere**
- **When the point P is on the sphere**
- **When the point P is inside the sphere**

(ii) Electric Field due to a charged spherical shell

Case(i) When the point P is out side the sphere*($r > r_0$)*

- Spherical shell of radius r_0, carrying a charge Q with centre O
- Imagine a concentric spherical shell of radius r as a Gaussian surface
- Electic field E_o at each point of surface is same & directed outward
- Let the electric field at the surface be E_o
- Net Electric flux through the whole surface



$$\phi_E = \oint_A E_o \cdot d\underline{A} = E_o (4\pi r^2)$$

By Gauss theorem
$$\oint_A E_o \cdot d\underline{A} = \frac{Q_{encl}}{\varepsilon_0}$$

Charge enclosed by surface $Q_{encl} = Q$

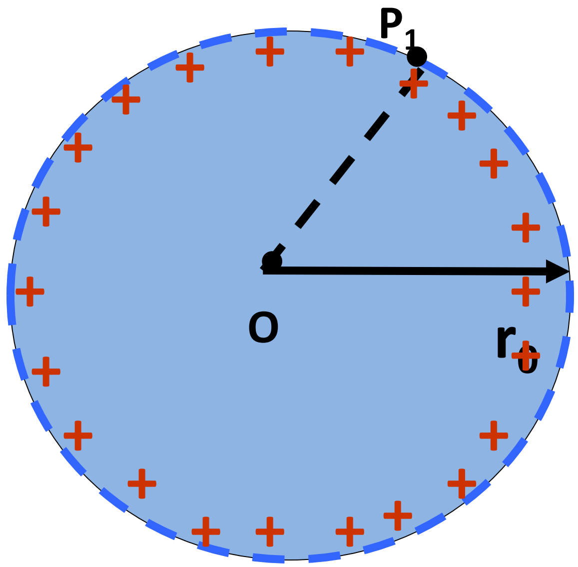
$$E_o = \frac{Q}{4\pi\varepsilon_0 r^2} \quad \text{N/C}$$

Hence the electric field strength at any pt outside a charged spherical shell is same as through the charge were placed at the centre O.

Case(ii) When the point P is on the sphere *($r = r_0$)*

If E_i is the electric field on the shell, then by symmetry E_i is same at each point of spherical surface and is directed outward

$$\oint_A E_i \cdot d\underline{A} = E_i (4\pi r^2)$$



Net charge enclosed by spherical surface $Q_{encl} = Q$

By Gauss theorem

$$\oint_A E_i \cdot d\underline{A} = \frac{Q_{encl}}{\varepsilon_0}$$

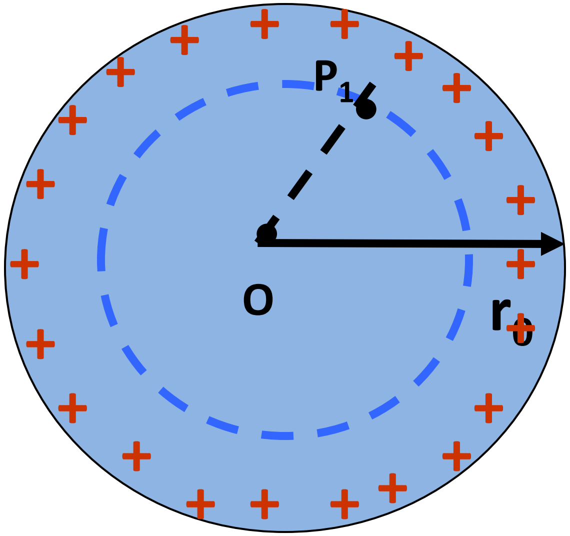
$$E_o = \frac{Q}{4\pi\varepsilon_0 r_0^{\ 2}} \qquad \text{N/C}$$

Case(iii) When the point P is inside the sphere *($r < r_0$)*

If E_i is the electric field inside the shell, then by symmetry E_i is same at each point of spherical surface and is directed outward

$$\oint_A E_i \cdot d\underline{A} = E_i(4\pi r^2)$$

Net charge enclosed by spherical surface $Q_{encl} = 0$



By Gauss theorem

$$\oint_A E_i \cdot d\underline{A} = \frac{Q_{encl}}{\varepsilon_0}$$

$$E_i 4\pi\, r^2 = \frac{1}{\varepsilon_0} \times 0$$

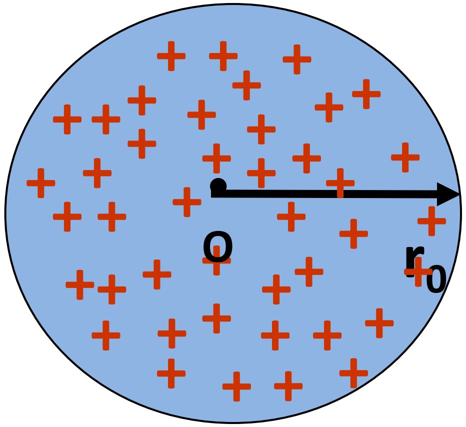
$$E_i = 0$$

Thus electric field strength at each point within the shell is zero

(ii) Electric Field due to uniformly charged non-conducting solid sphere

- Let non- conducting solid sphere of radius r_0 is uniformly charged with a charge q with charge density $\rho = \dfrac{q}{\frac{4}{3}\pi r^3}$

- We are to find the electric field intensity at a point P, at distance r from the center of the sphere in the following three cases.



➢ **When the point P is out side the solid sphere**

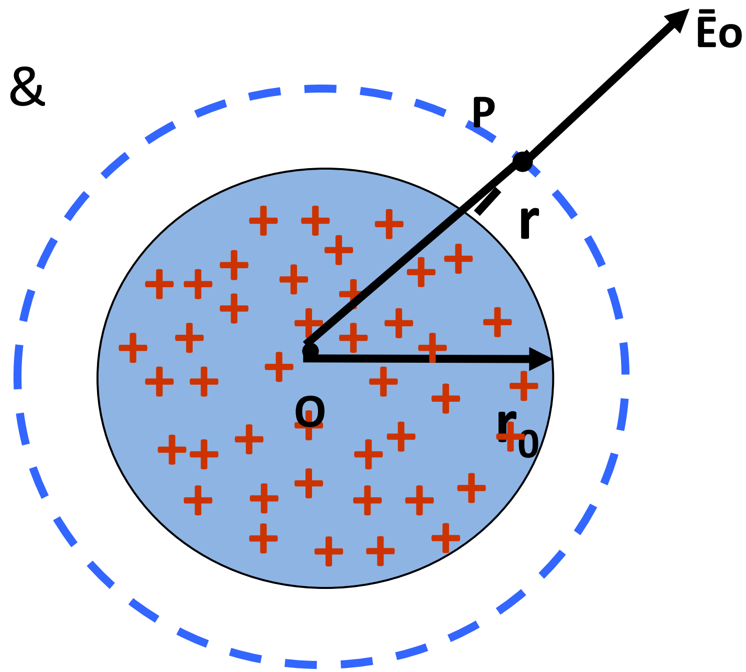
➢ **When the point P is on the solid sphere**

➢ **When the point P is inside the solid sphere** *(ii)* **Electric Field due to uniformly charged non-conducting solid sphere**

Case(i) When the point P is out side the solid sphere*($r > r_0$)*

- Spherical charge distribution of radius r_0, carrying a charge Q with centre O
- Imagine a spherical surface of radius r concentric with the spherical charge as a Gaussian surface
- Electic field E_o at each point of surface is same & directed outward
- Let the electric field at the surface be E_o
- Net Electric flux through the whole surface

$$\phi_E = \oint_A E_o \cdot dA = E_o (4\pi r^2)$$



By Gauss theorem the total charge enclosed by the spherical surface = Q

$$\oint_A E_o \cdot d\underline{A} = \frac{Q_{encl}}{\varepsilon_0}$$

Charge enclosed by surface Q_{encl} = Q

$$E_o = \frac{Q}{4\pi\varepsilon_0 r^2} \quad \text{N/C}$$

$E_i \,\alpha\, 1/r^2$

Hence the electric field strength at any pt outside a spherical charge distribution is the same as through the whole charge were concentrated at the center.

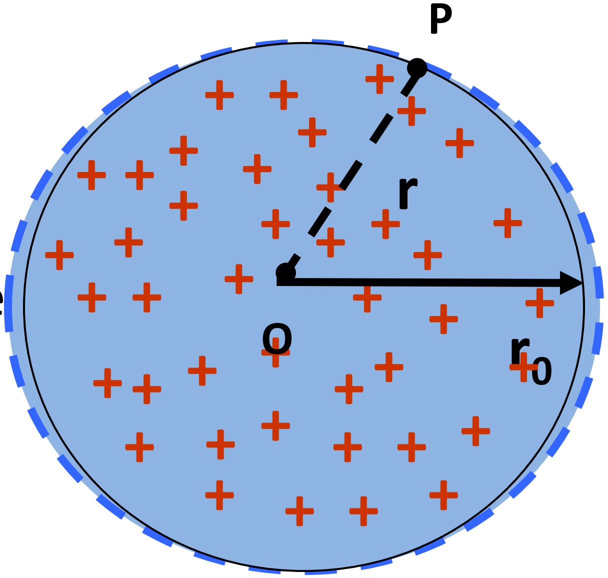
Case(ii) When the point P is on the sphere *($r = r_0$)*

- In this case the distance of point P from the center of the charge distribution is equal to its radius

- Electric flux through the whole surface

Charge enclosed by surface $Q_{encl} = Q$

$$\oint_A E \cdot d\underline{A} = E_o(4\pi r_0^{\;2})$$



By Gauss theorem the total charge enclosed by the spherical surface = Q

$$\oint_A E \cdot d\underline{A} = \frac{Q_{encl}}{\varepsilon_0}$$

Electric Field strength on the surface of the spherical charge distribution

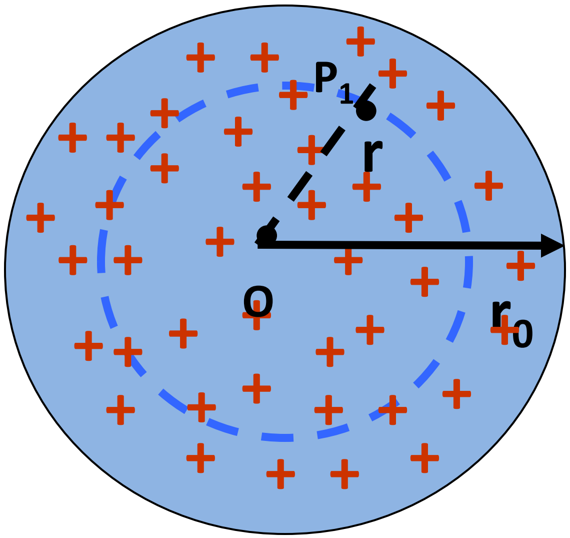
$$E = \frac{Q}{4\pi\varepsilon_0 r_0^{\;2}} \text{ N/C}$$

$E_i \,\alpha\, 1/r_0^{\;2}$

Case(iii) When the point P is inside the sphere *($r < r_0$)*

•Consider a spherical surface of radius r concentric with spherical charge as the Gaussian surface.

•Let ρ be the volume charge density (charge per unit volume) of uniform distribution of spherical charge



$$\rho = \frac{charge}{volume} = \frac{Q}{\frac{4}{3}\pi r_0^3} \Rightarrow Q = \frac{4}{3}\pi r_0^3 \rho$$

•Total electric flux through the whole surface

$$\oint_A E_i \cdot d\underline{A} = E_i(4\pi r^2)$$

Net charge enclosed by the Gaussian surface = $\frac{4}{3}\pi r^3 \rho$

By Gauss theorem $$\oint_A E_i \cdot d\underline{A} = \frac{4}{3}\frac{\pi r^3 \rho}{\varepsilon_0}$$

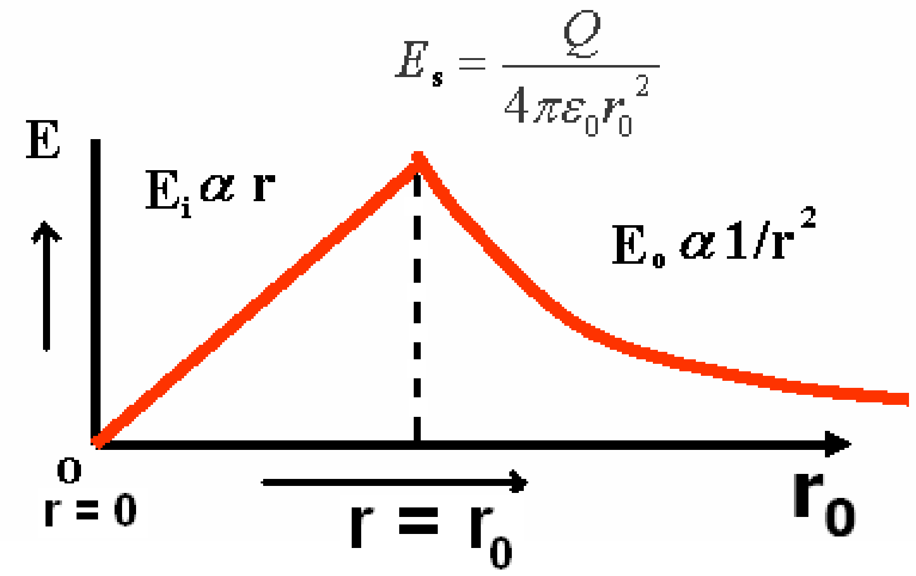
$$E_i = \frac{1}{4\pi\varepsilon_0 r^2}\left(\frac{4}{3}\pi r^3 \bullet \frac{Q}{\frac{4}{3}\pi r_0^{\ 3}}\right)$$

$$E_i = \frac{1}{4\pi\varepsilon_0}\frac{Qr}{r_0 3} \quad \text{N/C}$$

$$E_i \,\alpha\, r$$

- At the center of sphere r= 0 $\therefore$ E=0

•Thus the field strength is maximum at the surface of the spherical charge equal to $\frac{Q}{4\pi\varepsilon_0 r_0^{\ 2}}$



The variation of electric field strength with the distance from the center of spherical symmetric charge distribution