Electromagnetism

Lecture 4

Integral Calculus

- **Line (Path) Integrals**
- **Surface (Flux) Integrals**
- **Volume Integrals**

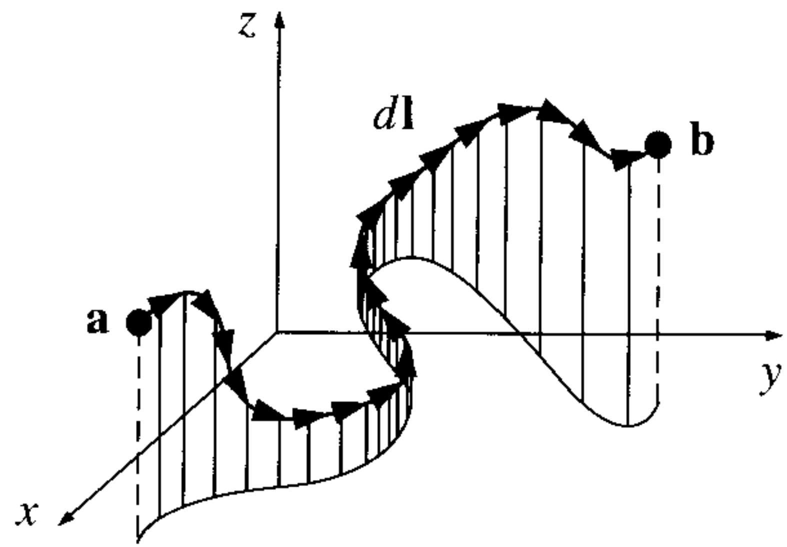
Line (Path) Integrals:

Let v - is a *vector function*

dl - is the infinitesimal displacement vector

Integral carried out along a *specified path from point a to point b* gives line or path integral.

$$\int_L V \bullet dl = \int_a^b V \bullet dl = \int_a^b |V| dl \cos\theta$$



Line integral depends critically on the particular path taken from a to b.

If path forms a closed loop i.e. a = b, the integral can be written as:

$$\int V \bullet dl$$

There is a **special type of vector which does not depend on path,** but depends on the co-ordinate s of the points a and point b. **The vector field is called as *conservative field*.**

The line integral of the ***conservative field*** along a closed path is ***zero*****.**

$$\oint V \bullet dl = 0$$

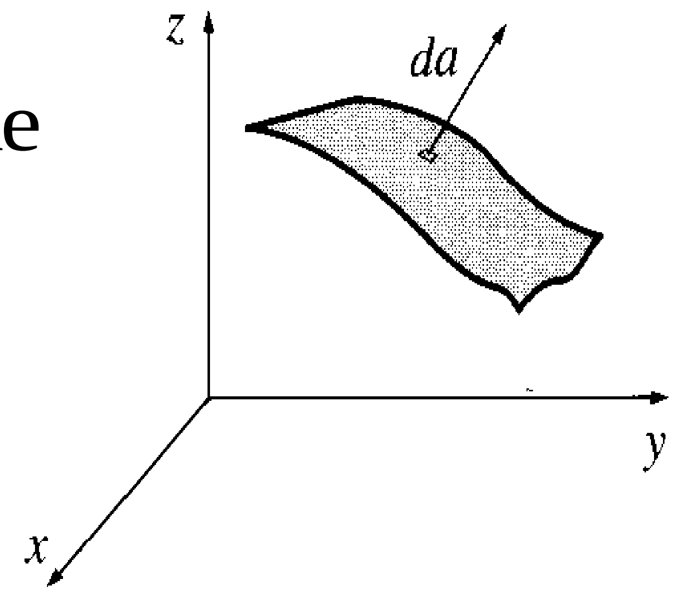
Surface (Flux) Integrals:

Let v - is a vector function

da - is an infinitesimal patch of area (direction is ⊥ to the surface)

If v is flow of a fluid (mass per unit area per unit time)

$$\int_s V \bullet da : \text{total mass per unit time passing through the surface}$$



If the surface is closed: $$\oint V \bullet da$$

- **For closed surface, direction of da is outward**

- **For open surfaces it's arbitrary.**

- **Surface integral depends on the particular surface chosen.**

- **There is a special class of vector function for which integral is *independent* of the surface, and is determined entirely by the *boundary line*.**

Volume Integrals:

The volume integrals as applied to closed volumes are of two type

- $Q = \int_v \rho_v dV$, where scalar ρ_v is the volume density function such as charge density function and Q is the total ccharge contained in the volume
- $\overline{P} = \int_v P dV$, this is the volume integral of a vector

In Cartesian coordinates, it can be written as

$$\overline{P} = a_x \int_v P_x dV + a_y \int_v P_y dV + a_z \int_v P_z dV$$

Gauss Divergence Theorem

According to divergence theorem:

The volume integral of divergence of a vector field A over a volume V is equal to the surface integral of that vector field A taken over the surface which encloses that volume V i.e,

$$\Rightarrow \quad \oint_S \vec{A}.\mathrm{d}\vec{S} = \underbrace{\int_V \vec{\nabla}.\vec{A}\,\mathrm{d}V}$$

This theorem is used to convert the volume integral into the surface integral or to convert the surface integral into the volume integral.

Example 6

Show that the divergence theorem holds for the vector field $\mathbf{A} = \mathbf{a}_r / r$ when the surface is that of a sphere of radius a centered at the origin. We have $\nabla \cdot \mathbf{A} = 1/r^2$ and

$$\oint_S \vec{A}.\mathrm{d}\vec{S} = \int_V \vec{\nabla}.\vec{A}\,\mathrm{d}V$$

$$\int_{\theta=0}^{\pi}\int_{\phi=0}^{2\pi} \frac{1}{a}\mathbf{a}_r \cdot \mathbf{a}_r a^2 \sin\theta\, d\theta d\phi = \int_{r=0}^{a}\int_{\theta=0}^{\pi}\int_{\phi=0}^{2\pi} \sin\theta\, dr\, d\theta d\phi$$

$$4\pi a = 4\pi a.$$

Stokes' Theorem

According to divergence theorem:

The line integral of a vector field A along the boundary of a closed curve C is equal to the surface integral of curl of that vector field when the surface integration is done over a surface S enclosed by the boundary C i.e.

$$\oint_C \mathbf{A} \cdot d\mathbf{l} = \iint_S (\nabla \times \mathbf{A}) \cdot d\mathbf{s}$$

This theorem is used to convert the surface integral into the line integral or to convert the line integral into the surface integral.

Example 7

Consider the portion of a sphere The surface is specified by $r = 4$, $0 \le \theta \le 0.1\pi$, $0 \le \phi \le 0.3\pi$, and the closed path forming its perimeter is composed of three circular arcs. We are given the field $\mathbf{H} = 6r\sin\phi\mathbf{a}_r + 18r\sin\theta\cos\phi\mathbf{a}_\phi$ and are asked to evaluate each side of Stokes' theorem.

$$\nabla \times \mathbf{H} = \frac{1}{r\sin\theta}(36r\sin\theta\cos\theta\cos\phi)\mathbf{a}_r + \frac{1}{r}\left(\frac{1}{\sin\theta}6r\cos\phi - 36r\sin\theta\cos\phi\right)\mathbf{a}_\theta$$

The differential path element $d\mathbf{L}$

$$d\mathbf{L} = dr\,\mathbf{a}_r + r\,d\theta\,\mathbf{a}_\theta + r\sin\theta\,d\phi\,\mathbf{a}_\phi$$

$$\oint \mathbf{H}\cdot d\mathbf{L} = \int(6r\sin\phi\mathbf{a}_r + 18r\sin\theta\cos\phi\mathbf{a}_\phi)\cdot[dr\,\mathbf{a}_r + r\,d\theta\,\mathbf{a}_\theta + r\sin\theta\,d\phi\,\mathbf{a}_\phi]$$

$$= \int 6r\sin\phi\,dr + 18r^2\sin^2\theta\cos\phi\,d\phi$$

For r = constant dr = 0

$$= \int_0^{0.3\pi} 18r^2\sin^2\theta\cos\phi\,d\phi$$

$$\oint \mathbf{H}\cdot d\mathbf{L} = \int_0^{0.3\pi}[18(4)\sin 0.1\pi\cos\phi]4\sin 0.1\pi\,d\phi = 288\sin^2 0.1\pi\sin 0.3\pi = 22.2\text{ A}$$

$$\nabla\times A = a_r \frac{1}{r\sin\theta}\left[\frac{\partial\left(A_\phi \sin\theta\right)}{\partial\theta} - \frac{\partial\left(A_\theta\right)}{\partial\phi}\right] + a_\theta \frac{1}{r}\left[\frac{1}{\sin\theta}\frac{\partial A_r}{\partial\phi} - \frac{\partial\left(rA_\phi\right)}{\partial r}\right] + a_\phi \frac{1}{r}\left[\frac{\partial\left(rA_\theta\right)}{\partial r} - \frac{\partial\left(A_r\right)}{\partial_\theta}\right]$$

$$\nabla \times \mathbf{H} = \frac{1}{r\sin\theta}(36r\sin\theta\cos\theta\cos\phi)\mathbf{a}_r + \frac{1}{r}\left(\frac{1}{\sin\theta}6r\cos\phi - 36r\sin\theta\cos\phi\right)\mathbf{a}_\theta$$

Since $d\mathbf{S} = r^2 \sin\theta \, d\theta \, d\phi \, \mathbf{a}_r$, the integral is

$$\int_S (\nabla \times \mathbf{H}) \cdot d\mathbf{S} = \int_0^{0.3\pi} \int_0^{0.1\pi} (36\cos\theta\cos\phi)16\sin\theta \, d\theta \, d\phi$$

$$= \int_0^{0.3\pi} 576(\tfrac{1}{2}\sin^2\theta)\Big|_0^{0.1\pi} \cos\phi \, d\phi$$

$$= 288\sin^2 0.1\pi \sin 0.3\pi = 22.2 \text{ A}$$

Thus, the results check Stokes' theorem.