# Electromagnetism

## Lecture 3

# OPERATOR

# DEL OPERATOR

- Written as $\nabla$ is the vector differential operator. Also known as the *gradient operator*. The operator in useful in defining:

$$\vec{\nabla} = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)$$

1. The gradient of a scalar V, written as $\nabla V$
2. The divergence of a vector **A**, written as $\nabla \bullet \mathbf{A}$
3. The curl of a vector **A**, written as $\nabla \times \mathbf{A}$
4. The Laplacian of a scalar V, written as $\nabla^2 V$

# Vector Derivatives

**First derivatives:**

Gradient ($\nabla$)

Divergence ($\nabla \bullet$)

Curl ($\nabla \times$)

**Second derivatives:**

The Laplacian ($\nabla^2$) and its relatives

# GRADIENT OF SCALAR

The result of applying the del-operator on a scalar function *V* is called the **gradient of *V***:

- G is the gradient of V. Thus

$$grad\ V = \nabla V = \frac{\partial V}{\partial x} a_x + \frac{\partial V}{\partial y} a_y + \frac{\partial V}{\partial z} a_z$$

- In cylindrical coordinates,

$$\nabla V = \frac{\partial V}{\partial r} a_r + \frac{1}{r} \frac{\partial V}{\partial \phi} a_\phi + \frac{\partial V}{\partial z} a_z$$

- In spherical coordinates,

$$\nabla V = \frac{\partial V}{\partial r} a_r + \frac{1}{r} \frac{\partial V}{\partial \theta} a_\theta + \frac{1}{r \sin \theta} \frac{\partial V}{\partial \phi} a_\phi$$

# Physical Interpretation

## Gradient : Maximum space rate change

- If the scalar function $\varphi$ represents the temperature, Then, $\nabla\phi$ = grad $\phi$ is temperature gradient or rate of change of temperature with distance

temperature $\phi$ = Magnitude

$\nabla\phi$ = Magnitude and direction

- Let V represent the potential function then $-\nabla V$ will represent the rate of change of potential with distance.

$$-\nabla V = \bar{E}$$

# EXAMPLE

Given a potential function $V = 2x^2 + 4y$ V in free space find the electric field at the origin.

$$\bar{E} = -\nabla V$$

$$\vec{E} = -\left[\left(\frac{\partial V}{\partial x}\right)\hat{a}_x + \left(\frac{\partial V}{\partial y}\right)\hat{a}_y + \left(\frac{\partial V}{\partial z}\right)\hat{a}_z\right]$$

$$\vec{E} = -\left[4x\,\hat{a}_x + 4\hat{a}_y\right] \quad \text{V/m}$$

At origin

$$\bar{E} = -4\,a\hat{y} \quad \text{V/m}$$

# EXAMPLE

Given : $U = r^2 z\cos 2\phi$

$$\nabla U = \frac{\partial U}{\partial r}\mathbf{a}_r + \frac{\partial U}{r\partial \phi}\mathbf{a}_\phi + \frac{\partial U}{\partial Z}\mathbf{a}_z$$

$$2rz\cos 2\phi\ \mathbf{a}_r - 2rz\sin 2\phi\ \mathbf{a}_\phi + r^2\cos 2\phi\ \mathbf{a}_z$$

Given : $W = 10R\sin^2\theta\cos\phi$

$$\nabla W = \frac{\partial W}{\partial R}\mathbf{a}_R + \frac{\partial W}{R\partial \theta}\mathbf{a}_\theta + \frac{\partial W}{R\sin\theta\partial \phi}\mathbf{a}_\phi$$

$$= 10\sin^2\theta\cos\phi\ \mathbf{a}_R + 10\sin 2\theta\cos\phi\ \mathbf{a}_\theta - 10\sin\theta\sin\phi\mathbf{a}_\phi$$

# The Divergence

The scalar product of the del-operator and a vector function is called the **divergence** of the vector function:

$$\vec{\nabla} \bullet \vec{A} = \left( \hat{\boldsymbol{a}}_x \frac{\partial}{\partial x} + \hat{\boldsymbol{a}}_y \frac{\partial}{\partial y} + \hat{\boldsymbol{a}}_z \frac{\partial}{\partial z} \right) \bullet \left( A_x \hat{\boldsymbol{a}}_x + A_y \hat{\boldsymbol{a}}_y + A_z \hat{\boldsymbol{a}}_z \right) = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$$

The divergence of a vector function is a scalar!

**What is the divergence?** Roughly speaking, the divergence is a measure of how fast the field lines stretch and/or spread out.

If two objects following the direction specified by the vector function increase their separation, the divergence of the vector function is positive. If their separation decreases, the divergence is negative.

# DIVERGENCE

- **In Cartesian coordinates,**

$$\nabla \bullet \mathbf{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$$

- **In cylindrical coordinates,**

$$\nabla \bullet \mathbf{A} = \frac{1}{r}\frac{\partial}{\partial r}(r A_r) + \frac{1}{r}\frac{\partial A_\phi}{\partial \phi} + \frac{\partial A_z}{\partial z}$$

- **In spherical coordinate,**

$$\nabla \bullet A = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 A_r) + \frac{1}{r \sin\theta}\frac{\partial}{\partial \theta}(A_\theta \sin\theta) + \frac{1}{r\sin\theta}\frac{\partial A_\phi}{\partial \phi}$$

# Physical Interpretation

**Divergence :**

**Rate of separation diverging or converging field**

Electric field density

$$\vec{D} = \left[ 10xyz\,\hat{a}_x + 5x^2 y\,\hat{a}_y \right]$$

Calculate charge density at (1,1,1)

$$\nabla \cdot \mathbf{D} = \rho_v$$

$$\rho_v = \left[ \left(\frac{\partial}{\partial x}\right)\hat{a}_x + \left(\frac{\partial}{\partial y}\right)\hat{a}_y + \left(\frac{\partial}{\partial z}\right)\hat{a}_z \right] \bullet \vec{D}$$

$$= \left[ \left(\frac{\partial}{\partial x} 10xyz\right) + \left(\frac{\partial}{\partial y} 5x^2 y\right) \right] \quad = 10\,yz + 5\,x^2$$

At (1,1,1) = 15 $c/m^3$ Diverge

At (0,0,0) = 0 $c/m^3$ neither diverge nor converge

At (1,-1,1) = - 5$c/m^3$ Converge

# The Curl

The curl of a vector function ***A*** is

$$\vec{\nabla} \times \vec{A} = \begin{vmatrix} \hat{\boldsymbol{a}}_x & \hat{\boldsymbol{a}}_y & \hat{\boldsymbol{a}}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix} = \left( \frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z} \right) \hat{\boldsymbol{a}}_x + \left( \frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x} \right) \hat{\boldsymbol{a}}_y + \left( \frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} \right) \hat{\boldsymbol{a}}_z$$

The curl of a vector function ***A*** is a vector.

Roughly speaking, the curl is a measure of how fast the field-lines of a vector field twist or bend in a direction set by the right-hand rule It is also denote the rate of rotation of field vector at particular point.

**The rotation will always be anticlockwise when the $\nabla \times \vec{A}$ is + ve**

**The rotation will always be clockwise when the $\nabla \times \vec{A}$ is – ve**

**There is no rotation is $\nabla \times \vec{A}$ is = 0**

# Physical Interpretation

# Curl : Rotation of field

$$\text{Given} A = \left[ -y\,\hat{\boldsymbol{a}}_x + x\,\hat{\boldsymbol{a}}_y \right].\ \text{Find the curl A ?}$$

$$\vec{\nabla} \times \vec{A} = \begin{vmatrix} \hat{\boldsymbol{a}}_x & \hat{\boldsymbol{a}}_y & \hat{\boldsymbol{a}}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix}$$

$$\vec{\nabla} \times \vec{A} = \left( \frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z} \right) \hat{\boldsymbol{a}}_x + \left( \frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x} \right) \hat{\boldsymbol{a}}_y + \left( \frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} \right) \hat{\boldsymbol{a}}_z$$

$$\vec{\nabla} \times \vec{A} = 2\,\hat{\boldsymbol{a}}_z$$

This function has a + ve curl so rotation will be anticlockwise.

# CURL OF VECTOR

- In Cartesian coordinates,

$$\nabla \times A = \begin{vmatrix} \mathbf{a}_x & \mathbf{a}_y & \mathbf{a}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix}$$

- In cylindrical coordinates,

$$\nabla \times \mathbf{A} = \frac{1}{r} \begin{vmatrix} a_r & r a_\phi & a_z \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \phi} & \frac{\partial}{\partial z} \\ A_r & r A_\phi & A_z \end{vmatrix}$$

# CURL OF VECTOR

- **In spherical coordinates,**

$$\nabla \times \mathbf{A} = \frac{1}{r^2 \sin\theta} \begin{vmatrix} a_r & r a_\theta & (r\sin\theta) a_\phi \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial \phi} \\ A_r & r A_\theta & (r\sin\theta) A_\phi \end{vmatrix}$$

# LAPLACIAN OF A SCALAR

- **The Laplacian of a scalar field V, written as $\nabla^2 V$ is defined as the divergence of the gradient of V.**

- **In Cartesian coordinates,**

$$\nabla^2 V = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2}$$

- In cylindrical coordinates,

$$\nabla^2 V = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial V}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 V}{\partial \phi^2} + \frac{\partial^2 V}{\partial z^2}$$

# LAPLACIAN OF A SCALAR

- In spherical coordinates,

$$\nabla^2 V = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial V}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial V}{\partial \theta}\right) + \frac{1}{r^2 \sin^2\theta}\frac{\partial^2 V}{\partial \phi^2}$$