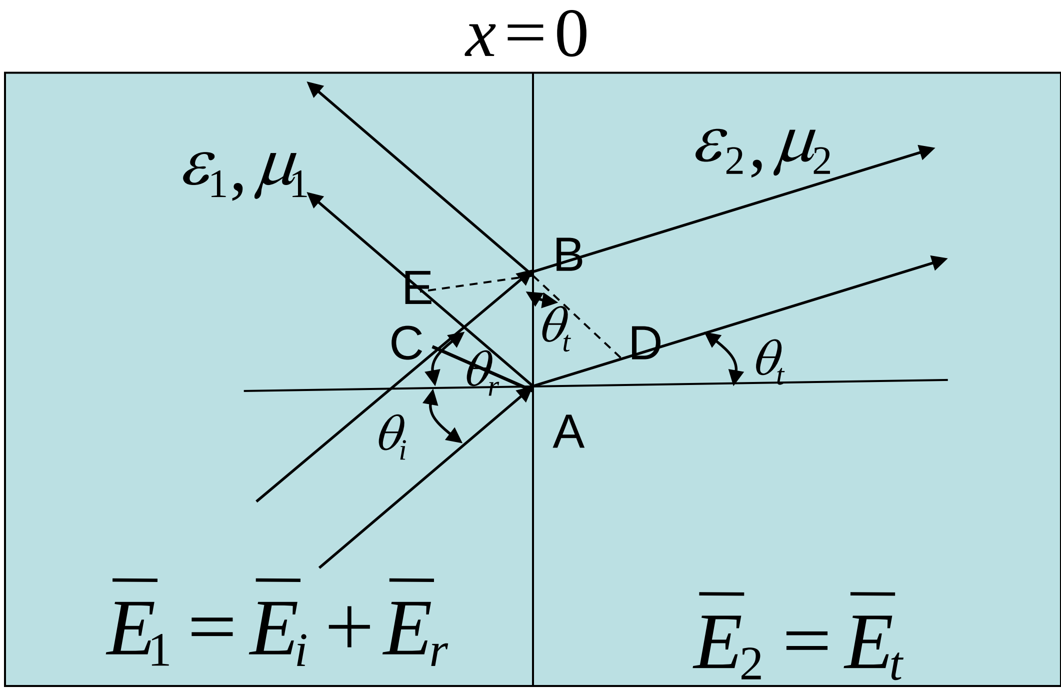
Boundary Conditions

Lecture 14

Oblique Incidence

- When a plane wave is incident upon a boundary surface that is not parallel to the plane, then a part of incident wave will reflected at an angle **θ**t and moves into the 2nd medium.

- Rest is reflected at an angle **θ**r into the same medium.



- The incident wave travels the distance CB.

- Refracted wave travels the distance AD.

- Reflected wave travels the distance AE.

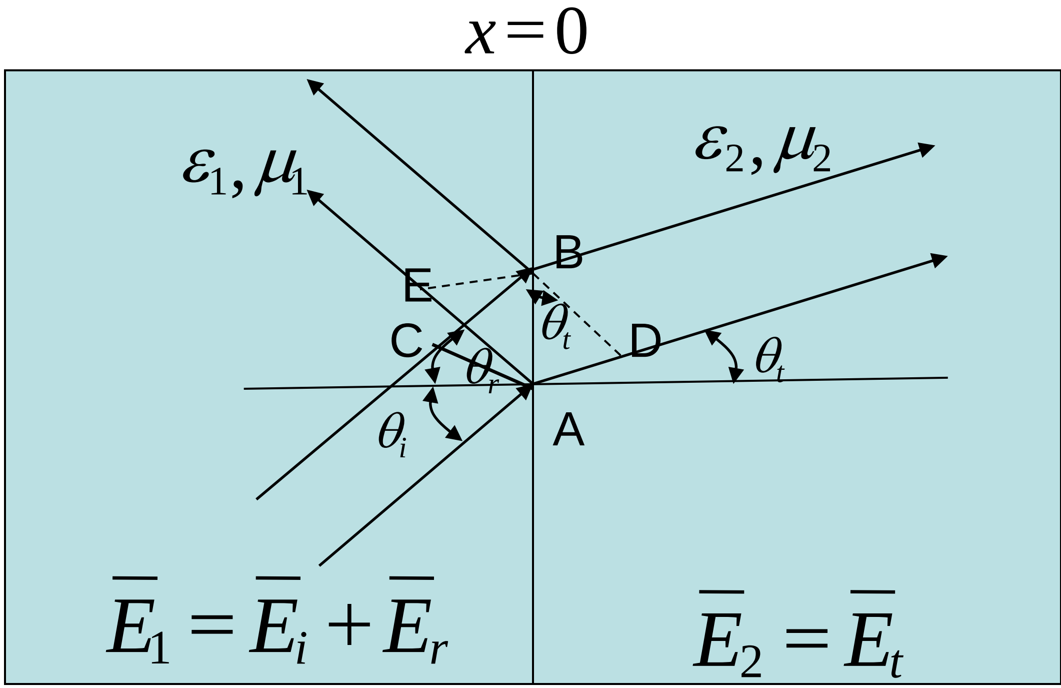
- If v1 and v2 be the speed of wave in medium 1 and 2 respectively

$$CB = v_1 t \text{ and } AD = v_2 t$$

$$\frac{CB}{AD} = \frac{v_1}{v_2}$$

by the help of geometry

$$\angle CAB = \theta i \text{ and } \angle ABD = \theta t$$



$$\therefore \sin\theta i = \frac{CB}{AB} \quad \text{and} \quad \sin\theta t = \frac{AD}{AB}$$

$$\frac{\sin\theta i}{\sin\theta t} = \frac{CB}{AD} = \frac{v1}{v2} \qquad \ldots 1$$

- If $\mu 1$ and $\mu 2$ are the permeabilities of the mediums the speed v1 and v2

$$v_1 = \frac{1}{\sqrt{\mu_1 \varepsilon_1}} \quad \text{and} \quad v_2 = \frac{1}{\sqrt{\mu_2 \varepsilon_2}}$$

•Putting the value of v1 and v2 in eq1

$$\frac{\sin\theta i}{\sin\theta t} = \frac{\sqrt{\mu_2\varepsilon_2}}{\sqrt{\mu_1\varepsilon_1}}$$

•For perfect dielectric μ1=μ2= μ

$$\frac{\sin\theta i}{\sin\theta t} = \frac{\sqrt{\varepsilon_2}}{\sqrt{\varepsilon_1}}$$

•Form fig AE = CB

$$\frac{AE}{AB} = \frac{CB}{AB}$$

$$\sin\theta_r = \sin\theta_i$$

$$\theta_r = \theta_i$$

$$\text{angle of reflection} = \text{angle of incidence}$$

•If n1 and n2 are the refractive indices of medium 1 and 2 respectively then

$$n_1 = \frac{c}{\mathrm{v}_1} \text{ and } n_2 = \frac{c}{\mathrm{v}_2}$$

$$\mathrm{v}_1 = \frac{c}{n_1} \text{ and } \mathrm{v}_2 = \frac{c}{n_2}$$

•Substituting v1 and v2 in eq 1

$$\frac{\sin\theta i}{\sin\theta t} = \frac{n_2}{n_1} \qquad ...2 \qquad \text{(Snell's law of refraction)}$$

•Thus concluded that em waves obey all the laws of reflection and refraction at a surface separating two dielectric media.

$$S = \frac{\vec{E}^2}{Z}\hat{n} \qquad \because \left(\vec{E} \cdot \hat{n} = 0, E \text{ being} \perp \text{to } \hat{n} \right)$$ Angle between E and n is 90^0

- Energy transmitted per square meter

$$S = \frac{E^2}{Z}$$

- Energy transmitted per square meter by incident wave

$$S_i = \frac{E_i^{\ 2} \cos \theta_i}{Z_1}$$

- Energy transmitted per square meter by reflected wave

$$S_r = \frac{E_r^{\ 2} \cos \theta_i}{Z_1}$$

- Energy transmitted per square meter by transmitted wave

$$S_t = \frac{E_t^{\ 2} \cos \theta_t}{Z_2}$$

•By the law of conservation of energy

Energy associated with the incident wave = Energy associated with the reflected wave+ Energy associated with the transmitted wave

$$\frac{E_i^{\,2}\cos\theta_i}{Z_1} = \frac{E_r^{\,2}\cos\theta_i}{Z_1} + \frac{E_t^{\,2}\cos\theta_t}{Z_2}$$

$$1 = \frac{E_r^{\,2}}{E_i^{\,2}} + \frac{E_t^{\,2}\cos\theta_t Z_1}{E_i^{\,2}\cos\theta_i Z_2}$$

$$\frac{E_r^{\,2}}{E_i^{\,2}} = 1 - \frac{E_t^{\,2}\cos\theta_t Z_1}{E_i^{\,2}\cos\theta_i Z_2} \qquad \left(\frac{Z_1}{Z_2} = \sqrt{\frac{\varepsilon_2}{\varepsilon_1}} \qquad \text{for perfect dielectric } \mu 1 = \mu 2 = \mu\right)$$

$$\frac{E_r^{\,2}}{E_i^{\,2}} = 1 - \frac{\sqrt{\varepsilon_2}\,E_t^{\,2}\cos\theta_t}{\sqrt{\varepsilon_1}\,E_i^{\,2}\cos\theta_i} \qquad \ldots 2$$

Oblique Incidence

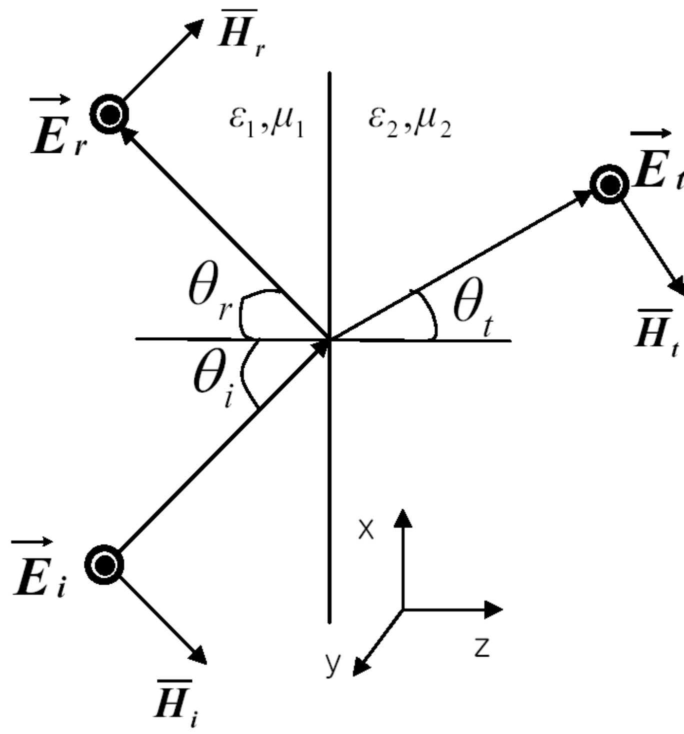
1. Incident plane wave is linearly polarized with its electric vector perpendicular to the plane of incident or parallel to the boundary surface.

2.Electric vector is parallel to the plane of incidence.

1.Electric vector is perpendicular to the plane of incidence

- **E electric vector lie normal to the plane of incidence**

- **H magnetic vector | | to the plane of incidence**



Applying boundary conditions, tangential components of E is continuous across the boundary

$$E_i + E_r = E_t$$

$$1 + \frac{E_r}{E_i} = \frac{E_t}{E_i} \qquad ...3$$

$$\left[\frac{E_r^{\ 2}}{E_i^{\ 2}} = 1 - \frac{\sqrt{\varepsilon_2} E_t^{\ 2} \cos\theta_t}{\sqrt{\varepsilon_1} E_i^{\ 2} \cos\theta_i} \right] \qquad ...2$$

Putting the value of E_t/E_i in eq 2

$$\frac{E_r^{\ 2}}{E_i^{\ 2}} = 1 - \frac{\sqrt{\varepsilon_2} \cos\theta_t}{\sqrt{\varepsilon_1} \cos\theta_i} \left(1 + \frac{E_r}{E_i}\right)^2$$

$$1-\frac{E_r{}^2}{E_i{}^2}=\frac{\sqrt{\varepsilon_2}\cos\theta_t}{\sqrt{\varepsilon_1}\cos\theta_i}\left(1+\frac{E_r}{E_i}\right)^2$$

$$\left(1+\frac{E_r}{E_i}\right)\left(1-\frac{E_r}{E_i}\right)=\frac{\sqrt{\varepsilon_2}\cos\theta_t}{\sqrt{\varepsilon_1}\cos\theta_i}\left(1+\frac{E_r}{E_i}\right)^2$$

$$\left(1-\frac{E_r}{E_i}\right)=\frac{\sqrt{\varepsilon_2}\cos\theta_t}{\sqrt{\varepsilon_1}\cos\theta_i}\left(1+\frac{E_r}{E_i}\right)$$

$$\sqrt{\varepsilon_1}\cos\theta_i\left(E_i-E_r\right)=\sqrt{\varepsilon_2}\cos\theta_t\left(E_i+E_r\right)$$

$$\left(\sqrt{\varepsilon_1}\cos\theta_i+\sqrt{\varepsilon_2}\cos\theta_t\right)E_r=\left(\sqrt{\varepsilon_1}\cos\theta_i-\sqrt{\varepsilon_2}\cos\theta_t\right)E_i$$

$$\frac{E_r}{E_i}=\frac{\sqrt{\varepsilon_1}\cos\theta_i-\sqrt{\varepsilon_2}\cos\theta_t}{\sqrt{\varepsilon_1}\cos\theta_i+\sqrt{\varepsilon_2}\cos\theta_t}=R_\perp$$

Reflection coefficient for horizontal polarization

Similarly the transmission coefficient for horizontal polarization

$$T_{\perp} = \frac{E_t}{E_i} = 1 + \frac{E_r}{E_i}$$

$$T_{\perp} = 1 + \frac{\sqrt{\varepsilon_1}\cos\theta_i - \sqrt{\varepsilon_2}\cos\theta_t}{\sqrt{\varepsilon_1}\cos\theta_i + \sqrt{\varepsilon_2}\cos\theta_t}$$

$$\boxed{T_{\perp} = \frac{2\sqrt{\varepsilon_1}\cos\theta_i}{\sqrt{\varepsilon_1}\cos\theta_i + \sqrt{\varepsilon_2}\cos\theta_t}}$$

But

$$\frac{\sin\theta_t}{\sin\theta_i} = \frac{\sqrt{\varepsilon_1}}{\sqrt{\varepsilon_2}} \Rightarrow \sqrt{\varepsilon_1} = \sqrt{\varepsilon_2}\,\frac{\sin\theta_t}{\sin\theta_i}$$

$$R_{\perp} = \frac{\sin\theta_t \cos\theta_i - \sin\theta_i \cos\theta_t}{\sin\theta_t \cos\theta_i + \sin\theta_i \cos\theta_t}$$

$$R_{\perp} = \frac{\sin(\theta_i - \theta_t)}{\sin(\theta_i + \theta_t)}$$

$$T_{\perp} = \frac{2\sin\theta_t \cos\theta_i}{\sin\theta_t \cos\theta_i + \sin\theta_i \cos\theta_t}$$

$$T_{\perp} = \frac{2\sin\theta_t \cos\theta_i}{\sin(\theta_i + \theta_t)}$$

1.Electric vector is parallel to the plane of incidence

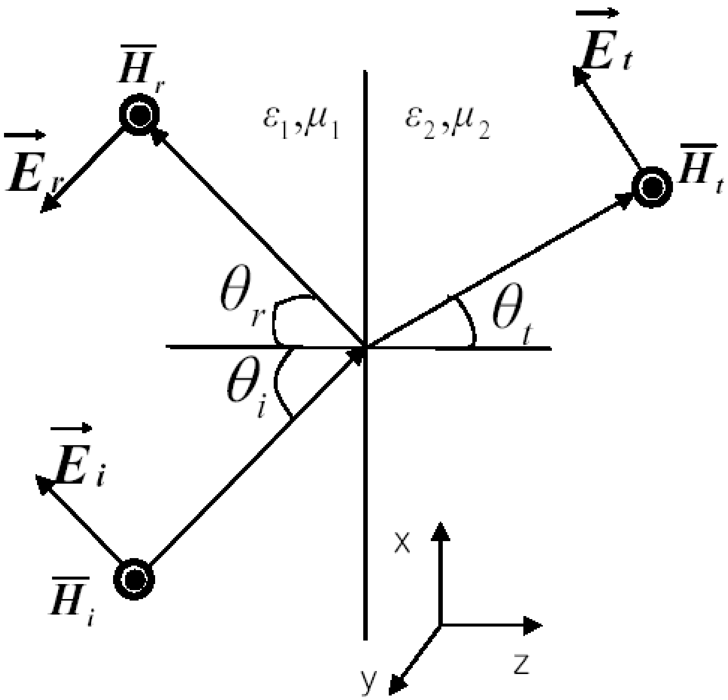
- **E electric vector lie in the plane of incidence**
- **H magnetic vector ⊥ to the plane of incidence**

Applying boundary conditions, tangential components of E is continuous across the boundary

$$E_i \cos\theta_i - E_r \cos\theta_i = E_t \cos\theta_t$$

$$1 - \frac{E_r}{E_i} = \frac{E_t}{E_i}\left(\frac{\cos\theta_t}{\cos\theta_i}\right)$$

$$\frac{E_t}{E_i} = 1 - \frac{E_r}{E_i}\left(\frac{\cos\theta_i}{\cos\theta_t}\right)$$



Putting the value of E_t/E_i in eq 2

$$\frac{E_r^{\,2}}{E_i^{\,2}} = 1 - \frac{\sqrt{\varepsilon_2}}{\sqrt{\varepsilon_1}}\left(1 - \frac{E_r}{E_i}\right)^2 \frac{\cos^2\theta_i \cos\theta_t}{\cos^2\theta_t \cos\theta_i}$$

$$\left[\frac{E_r^{\,2}}{E_i^{\,2}} = 1 - \frac{\sqrt{\varepsilon_2}\,E_t^{\,2}\cos\theta_t}{\sqrt{\varepsilon_1}\,E_i^{\,2}\cos\theta_i}\right] \quad ...2$$

$$1-\frac{E_r^{\ 2}}{E_i^{\ 2}}=\frac{\sqrt{\varepsilon_2}}{\sqrt{\varepsilon_1}}\left(1-\frac{E_r}{E_i}\right)^2\frac{\cos\theta_i}{\cos\theta_t}$$

$$\left(1+\frac{E_r}{E_i}\right)\left(1-\frac{E_r}{E_i}\right)=\frac{\sqrt{\varepsilon_2}}{\sqrt{\varepsilon_1}}\left(1-\frac{E_r}{E_i}\right)^2\frac{\cos\theta_i}{\cos\theta_t}$$

$$\left(1+\frac{E_r}{E_i}\right)=\frac{\sqrt{\varepsilon_2}\cos\theta_i}{\sqrt{\varepsilon_1}\cos\theta_t}\left(1-\frac{E_r}{E_i}\right)$$

$$\sqrt{\varepsilon_1}\cos\theta_t\left(E_i+E_r\right)=\sqrt{\varepsilon_2}\cos\theta_i\left(E_i-E_r\right)$$

$$\left(\sqrt{\varepsilon_1}\cos\theta_t+\sqrt{\varepsilon_2}\cos\theta_i\right)E_r=\left(\sqrt{\varepsilon_2}\cos\theta_i-\sqrt{\varepsilon_1}\cos\theta_t\right)E_i$$

$$\frac{E_r}{E_i}=\frac{\sqrt{\varepsilon_2}\cos\theta_i-\sqrt{\varepsilon_1}\cos\theta_t}{\sqrt{\varepsilon_1}\cos\theta_t+\sqrt{\varepsilon_2}\cos\theta_i}=R_{\text{II}}$$

Reflection coefficient for vertical polarization

Similarly the transmission coefficient for verticalpolarization

$$T_{\perp} = \frac{E_t}{E_i} = 1 + \frac{E_r}{E_i}$$

$$T_{\perp} = 1 + \frac{\sqrt{\varepsilon_2}\cos\theta_i - \sqrt{\varepsilon_1}\cos\theta_t}{\sqrt{\varepsilon_1}\cos\theta_t + \sqrt{\varepsilon_2}\cos\theta_i}$$

$$\boxed{T_{\perp} = \frac{2\sqrt{\varepsilon_2}\cos\theta_i}{\sqrt{\varepsilon_1}\cos\theta_t + \sqrt{\varepsilon_2}\cos\theta_i}}$$

But

$$\frac{\sin\theta_t}{\sin\theta_i} = \frac{\sqrt{\varepsilon_1}}{\sqrt{\varepsilon_2}} \Rightarrow \sqrt{\varepsilon_1} = \sqrt{\varepsilon_2}\,\frac{\sin\theta_t}{\sin\theta_i}$$

$$R_{II} = \frac{\sin\theta_i \cos\theta_i - \sin\theta_t \cos\theta_t}{\sin\theta_t \cos\theta_t + \sin\theta_i \cos\theta_i}$$

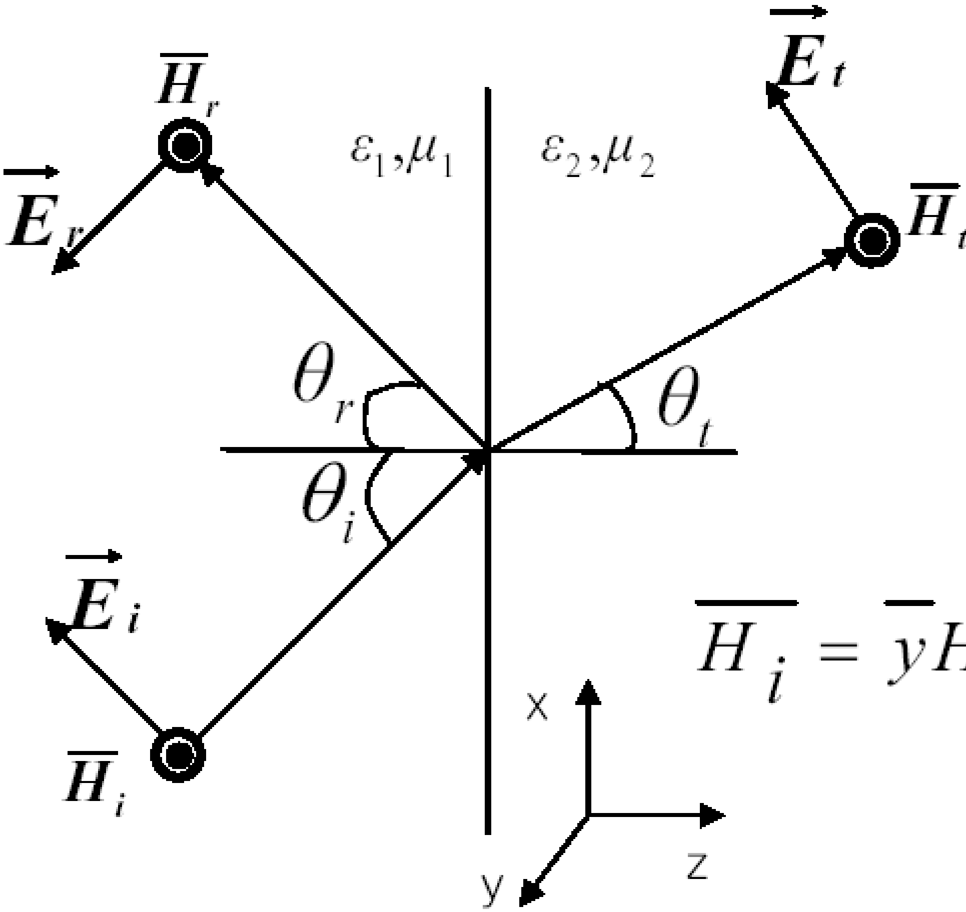
$$R_{II} = \frac{\sin 2\theta_i - \sin 2\theta_t}{\sin 2\theta_i + \sin 2\theta_t} = \frac{\tan(\theta_i - \theta_t)}{\tan(\theta_i + \theta_t)}$$

$$T_{II} = \frac{2\sin\theta_i \cos\theta_i}{\sin\theta_t \cos\theta t_i + \sin\theta_i \cos\theta_i}$$

$$T_{II} = \frac{2\sin\theta_i \cos\theta_i}{\sin 2\theta_i + \sin 2\theta_t}$$

$$T_{II} = \frac{2\sin\theta_i \cos\theta_i}{\sin(\theta_i + \theta_t)\cos(\theta_t - \theta_i)}$$

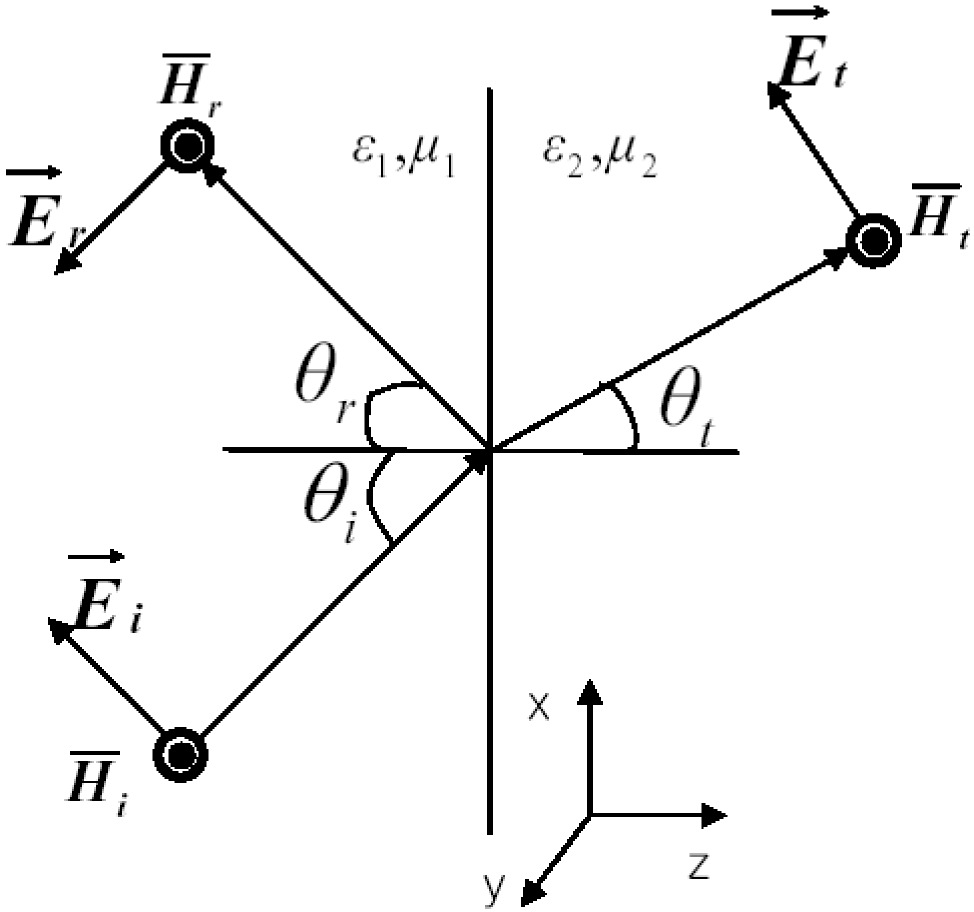
Parallel Polarization



$$\overline{H_i} = \overline{y} H_i \exp(-j\beta_{1x} x) \exp(-j\beta_{1z} z)$$

$$\overline{H_r} = \overline{y} H_r \exp(-j\beta_{rx} x) \exp(j\beta_{rz} z)$$

$$\overline{H_t} = \overline{y} H_t \exp(-j\beta_{2x} x) \exp(-j\beta_{2z} z)$$



$$E = H\eta$$

$$\overline{E_i} = (\overline{x}\cos\theta_i - \overline{z}\sin\theta_i)H_i\eta_1 \exp(-j\beta_{1x}x)\exp(-j\beta_{1z}z)$$

$$\overline{E_r} = (-\overline{x}\cos\theta_r - \overline{z}\sin\theta_r)H_r\eta_1 \exp(-j\beta_{rx}x)\exp(j\beta_{rz}z)$$

$$\overline{E_t} = (\overline{x}\cos\theta_t - \overline{z}\sin\theta_t)H_t\eta_2 \exp(-j\beta_{2x}x)\exp(-j\beta_{2z}z)$$

$$\frac{H_r}{H_i} = -\Gamma \qquad \frac{H_t}{H_i} = \frac{E_t / \eta_2}{E_i / \eta_1} = \tau \frac{\eta_1}{\eta_2}$$

B.C.'s at z=0

1) $\bar{H}_{\tan}$ coutinuous $(\bar{H}_i + \bar{H}_r = \bar{H}_t)$

$$\overline{H_i} = \bar{y} H_i \exp(-j\beta_{1x} x) \exp(-j\beta_{1z} z)$$

$$\overline{H_r} = \bar{y} H_r \exp(-j\beta_{rx} x) \exp(j\beta_{rz} z)$$

$$\overline{H_t} = \bar{y} H_t \exp(-j\beta_{2x} x) \exp(-j\beta_{2z} z)$$

$$\exp(-j\beta_{1x}x) - \Gamma\exp(-j\beta_{rx}x) = \frac{\eta_1}{\eta_2}\tau\exp(-j\beta_{2x}x)$$

$$\beta_{1x} = \beta_{rx} = \beta_{2x}$$

$$1 - \Gamma = \frac{\eta_1}{\eta_2}\tau \qquad \text{.....(A)}$$

B.C.'s at $z=0$

2) $\bar{E}_{tan}$ continuous

$$\cos\theta_i + \Gamma\cos\theta_i = \tau\cos\theta_t \qquad \text{........(B)}$$

reflection coefficient

$$\Gamma_{\parallel} = \frac{\eta_2 \cos\theta_t - \eta_1 \cos\theta_i}{\eta_2 \cos\theta_t + \eta_1 \cos\theta_i}$$

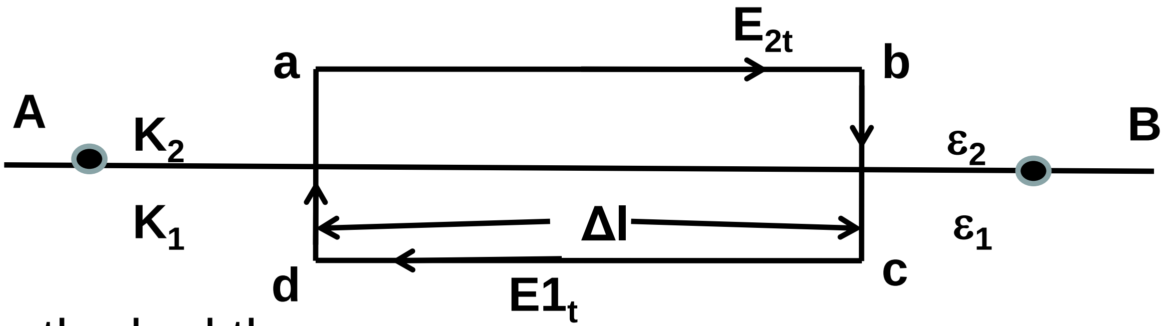
transmission coefficient

$$\tau_{\parallel} = \frac{2\eta_2 \cos\theta_i}{\eta_2 \cos\theta_t + \eta_1 \cos\theta_i}$$

Electrostatic Boundary Conditions

Consider a rectangular closed path abcda of infinitesimal area in the plane normal to the boundary and with its side ab and cd parallel to and on either side of th boundary.

Since the line integral ∫ E.dl for closed path is zero.



Here considering the closed path abcd then

$$\oint E \cdot dl = \int_a^b E \cdot dl + \int_b^c E \cdot dl + \int_c^d E \cdot dl + \int_d^a E \cdot dl$$

if the loop is shrunk by letting $h \rightarrow 0$

then the parts bc and ad are negligible

$$\oint E \cdot dl = \int_a^b E_2 \cdot dl + \int_c^d E_1 \cdot dl = 0$$

E_{1t} and E_{2t} are components of electric field intensities then

$$\oint E \cdot dl = E_{1t}\Delta l - E_{2t}\Delta l = 0$$

$$E_{1t} = E_{2t}$$

Hence any point on the boundary the components of E1 and E2 tangential to the boundary are equal

If medium 1 is conductor then E_{1t} =0 then

$$E_{2t} = 0$$

Boundary Conditions for Displacement vector D

Consider a cylindrical bax ABCD which intersect the two dielectric media having permittivities ε_1 and ε_2.

Supposing dS is the area of each plane and which is also the area of the boundary enclosed by the cylinder.

Now the charge enclosed by cylinder will be $\boldsymbol{\rho} = \sigma ds$

For finding the boundary conditions for D, apply Gauss's law
Here cylinder will behave like Gaussian surface.

$$\oint D \cdot ds = \sigma ds$$

$$= \int_{upper} D.ds + \int_{lower} D.ds + \int_{curved} D.ds$$

Let h$\longrightarrow 0$ then the curved path vanishes

$$\int_{upper} D.ds + \int_{lower} D.ds = \int_{s} \sigma ds$$