Boundary Conditions

Lecture 13

Boundary Conditions

- Boundary conditions are the relations between the electromagnetic field vectors on two sides of the interface that separates the two media.

- These conditions are dictated by types of material by which media are made of.

Electrostatic Boundary Conditions

- **At any point on the boundary,**
  - the components of $\boldsymbol{E_1}$ and $\boldsymbol{E_2}$ tangential to the boundary are equal

$$\boxed{E_{1t} = E_{2t}}$$

  - the components of $\boldsymbol{D_1}$ and $\boldsymbol{D_2}$ normal to the boundary are equal

$$\boxed{D_{1n} = D_{2n}}$$

Magnetic Boundary Conditions

- The normal component of $\boldsymbol{B_1}$ and $\boldsymbol{B_2}$ continuous across a interface:

$$\boxed{B_{1n} = B_{2n}}$$

- The tangential component of a $\boldsymbol{H_1}$ and $\boldsymbol{H_2}$ to the boundary are equal

$$\boxed{H_{1t} = H_{2t}}$$

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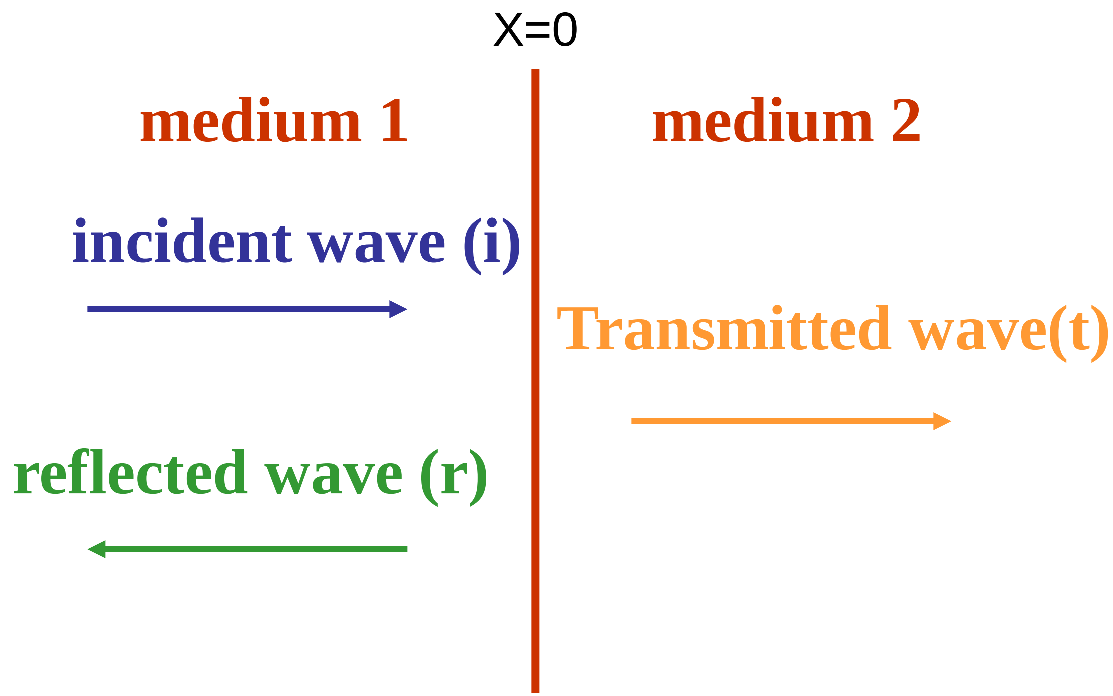
Reflection and Transmission of Waves at Planar Interfaces

1. Reflection of uniform plane waves by perfect dielectric for normal incidence.



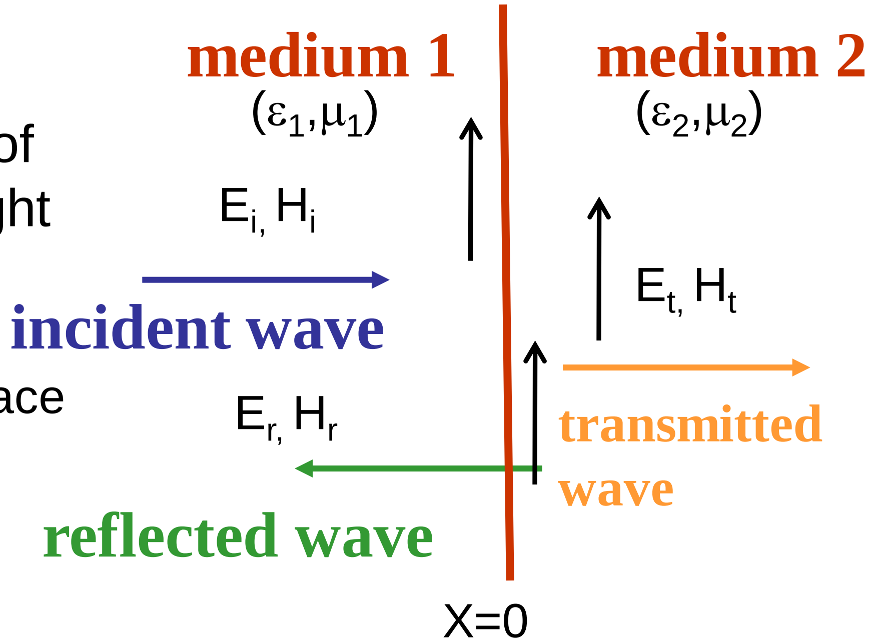
A traveling E wave approaches the interface x=0 from region 1,x<0.

Ei and Er are at x=-0, while Et is at x=+0

Consider the case of plane em wave travelling in x direction and polarized in y direction and is incident normally an the interface from left.

It give rise to a reflected wave which travels back to the left in medium 1 and transmitted wave which continues on the right in medium 2.

Let (ε_1,μ_1) and (ε_2,μ_2) are the dielectric permittivities and magnetic permeabilities of two medium 1 and medium 2 at left and right of the plane x=0.



In this case E and H are tangential to the interface

The intrinsic impedance for medium 1

$$Z_1 = \sqrt{\frac{\mu_1}{\varepsilon_1}}$$

The intrinsic impedance for medium 2

$$Z_2 = \sqrt{\frac{\mu_2}{\varepsilon_2}}$$

The electric ad magnetic fields associated with incident, reflected and transmitted em waves are related as

$$E_i = Z_1 H_i \qquad ...1$$

$$E_r = -Z_1 H_r \qquad ...2$$

$$E_t = Z_2 H_t \qquad ...3$$

According to the boundary conditions at the interface for the continuity of tangential components of E and H, the electric and magnetic fields on each side of the boundary are same.

$$E_i + E_r = E_t \qquad ...4$$

$$H_i + H_r = H_t \qquad ...5$$

from eq 1, 2 and 3

$$H_i = \frac{E_i}{Z_1}, \; H_r = -\frac{E_r}{Z_1}, \; H_t = \frac{E_t}{Z_2}$$

putting the value of H_i, H_r, H_t in eq 5

$$\frac{E_i}{Z_1} - \frac{E_r}{Z_1} = \frac{E_t}{Z_2} \qquad ...6$$

Putting the value of E_t from eq 4 in eq 6

$$\frac{1}{Z_1}\left(E_i - E_r\right) = \frac{\left(E_i + E_r\right)}{Z_2}$$

$$E_i\left(Z_2 - Z_1\right) = E_r\left(Z_2 + Z_1\right)$$

$$\frac{E_r}{E_i} = \frac{\left(Z_2 - Z_1\right)}{\left(Z_2 + Z_1\right)} = R_E = R\text{eflection cofficient}$$

$$\text{for electric field intensity} \quad ....7$$

$$\text{Transmission cofficient for electric field intensity}$$

$$T_E = \frac{E_t}{E_i} = \frac{E_i + E_r}{E_i}$$

$$= 1 + \frac{E_r}{E_i} = 1 + \frac{(Z_2 - Z_1)}{(Z_2 + Z_1)}$$

$$T_E = \frac{2Z_2}{(Z_2 + Z_1)} \qquad ...8$$

Similarly the reflection and transmission coefficient for magnetic field intensity H is obtained

$$R\text{eflection cofficient for magnetic field intensity}$$

$$R_H = \frac{H_r}{H_i} = -\frac{E_r}{E_i} = \frac{(Z_1 - Z_2)}{(Z_1 + Z_2)} \qquad ...9$$

Transmission cofficient for magnetic field intensity

$$T_H = \frac{H_t}{H_i} = \frac{E_t / Z_2}{E_i / Z_1} = \frac{Z_1 E_t}{Z_2 E_i} = \frac{Z_1}{Z_2} T_E$$

$$T_H = \frac{2 Z_1 Z_2}{Z_2 (Z_2 + Z_1)} = \frac{2 Z_1}{(Z_2 + Z_1)} \qquad ...10$$

Eq (7),(8),(9) and (10) are valid for a boundary between any two materials

In general equations for E can be written as

$$\overline{E}_i(x,t) = E_{i0} e^{-\gamma_1 x} e^{j\omega t} \hat{a}_x$$

$$\overline{E}_r(x,t) = E_{r0} e^{-\gamma_1 x} e^{j\omega t} \hat{a}_x$$

$$\overline{E}_t(x,t) = E_{t0} e^{-\gamma_2 x} e^{j\omega t} \hat{a}_x$$

Here E_{io}, E_{ro} and E_{to} are the electric field intensities at x=0 and γ_1, γ_2 are the propagation constants in 1[st] and 2[nd] medium

The equations for H can be written as

$$\overline{H}_i(x,t) = H_{i0} e^{-\gamma_1 x} e^{j\omega t} \hat{a}_y$$

$$\overline{H}_r(x,t) = H_{r0} e^{-\gamma_1 x} e^{j\omega t} \hat{a}_y$$

$$\overline{H}_t(x,t) = H_{t0} e^{-\gamma_2 x} e^{j\omega t} \hat{a}_y$$

Here H_{io}, H_{ro} and H_{to} are the magnetic field intensities at x=0

Intrinsic impedance for

for perfect dielectric

$$Z = \sqrt{\frac{\mu}{\varepsilon}}$$

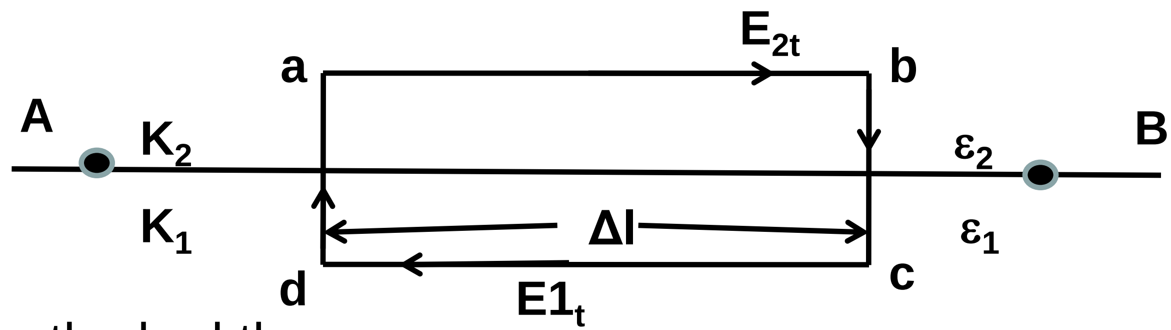
for free space

$$Z = \sqrt{\frac{\mu_0}{\varepsilon_0}}$$

Electrostatic Boundary Conditions

Consider a rectangular closed path abcda of infinitesimal area in the plane normal to the boundary and with its side ab and cd parallel to and on either side of th boundary.

Since the line integral ∫ E.dl for closed path is zero.



Here considering the closed path abcd then

$$\oint E\cdot dl = \int_a^b E\cdot dl + \int_b^c E\cdot dl + \int_c^d E\cdot dl + \int_d^a E\cdot dl$$

$$\text{if the loop is shrunk by letting } h \to 0$$

$$\text{then the parts bc and ad are negligible}$$

$$\oint E\cdot dl = \int_a^b E_2\cdot dl + \int_c^d E_1\cdot dl = 0$$

E_{1t} and E_{2t} are components of electric field intensities then

$$\oint E \cdot dl = E_{1t}\Delta l - E_{2t}\Delta l = 0$$

$$E_{1t} = E_{2t}$$

Hence any point on the boundary the components of E1 and E2 tangential to the boundary are equal

If medium 1 is conductor then E_{1t} =0 then

$$E_{2t} = 0$$

Boundary Conditions for Displacement vector D

Consider a cylindrical bax ABCD which intersect the two dielectric media having permittivities ε_1 and ε_2.

Supposing dS is the area of each plane and which is also the area of the boundary enclosed by the cylinder.

Now the charge enclosed by cylinder will be $\boldsymbol{\rho} = \sigma ds$

For finding the boundary conditions for D, apply Gauss's law
Here cylinder will behave like Gaussian surface.

$$\oint D \cdot ds = \sigma ds$$

$$= \int_{upper} D.ds + \int_{lower} D.ds + \int_{curved} D.ds$$

Let h$\longrightarrow 0$ then the curved path vanishes

$$\int_{upper} D.ds + \int_{lower} D.ds = \int_{s} \sigma ds$$