POISSON'S & LAPLACE'S EQUATION

$$\nabla^2 \Phi = -\rho_v / \varepsilon$$ *Poisson's equation*

$$\nabla^2 \Phi = 0$$ *Laplace's equation*

POISSON AND LAPLACE EQUATIONS

$$\mathbf{D} = \epsilon \mathbf{E}$$ ……(1)

$$\nabla \cdot \mathbf{D} = \rho$$ ……(2)

Substituting Equation 1 into 2

$$\nabla \cdot \epsilon \mathbf{E} = \rho$$ ……….. (3)

Since $\mathbf{E} = -\nabla\Phi$, the last result can be written

$$\nabla \cdot (\epsilon \nabla \Phi) = -\rho$$ ……..(4)

Now by vector identity $\nabla \bullet \left[a \nabla b \right] = \nabla a \bullet \nabla b + a \nabla^2 b$

$$\boldsymbol{\varepsilon}\nabla \cdot (\nabla\Phi) + \nabla\Phi \cdot \nabla\boldsymbol{\varepsilon} = -\rho$$

$$\boxed{\boldsymbol{\varepsilon}\nabla^2\Phi + \nabla\Phi \cdot \nabla\boldsymbol{\varepsilon} = -\rho} \quad \text{.......(5)}$$

In the special case where the dielectric is homogeneous, $\boldsymbol{\varepsilon}$ is not a function of position, $\nabla\boldsymbol{\varepsilon} = 0$, and Equation **(5) reduces to**

$$\boxed{\nabla^2\Phi = -\rho\, / \boldsymbol{\varepsilon}} \quad \textit{Poisson's equation}$$

For Charge Free region (ρ= 0) Poisson's equation reduces to

$$\boxed{\nabla^2\Phi = 0} \quad \textit{Laplace's equation}$$

Cartesian coordinate system

$$\nabla^2 V = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0$$

Cylindrical Coordinate System

$$\nabla^2 V = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial V}{\partial r}\right) + \frac{1}{r}\frac{\partial^2 V}{\partial \phi^2} + \frac{\partial^2 V}{\partial z^2} = 0$$

Spherical Coordinate System

$$\nabla^2 V = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial V}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial V}{\partial \theta}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial^2 V}{\partial \phi^2} = 0$$

Example:

Consider a parallel conductor where V = 0 at z = 0 and V = 100 Volts at z = d. Calculate potential as a Function of z.

$$\frac{\partial^2 V}{\partial x^2}+\frac{\partial^2 V}{\partial y^2}+\frac{\partial^2 V}{\partial z^2}=0$$

Since V is not the function of x and y so Laplace's equation reduces to

$$\frac{\partial^2 V}{\partial z^2}=0$$

V = A z +B

Using given Conditions V= 0 at z = 0 Provide B = 0
V =100 at z =d gives A = 100/d

V = 100(z/d) Volts

Two ways of calculating **B** produced by currents:

i) Biot-Savart Law: Field of a "current element" (analagous to a point charge in electrostatics).

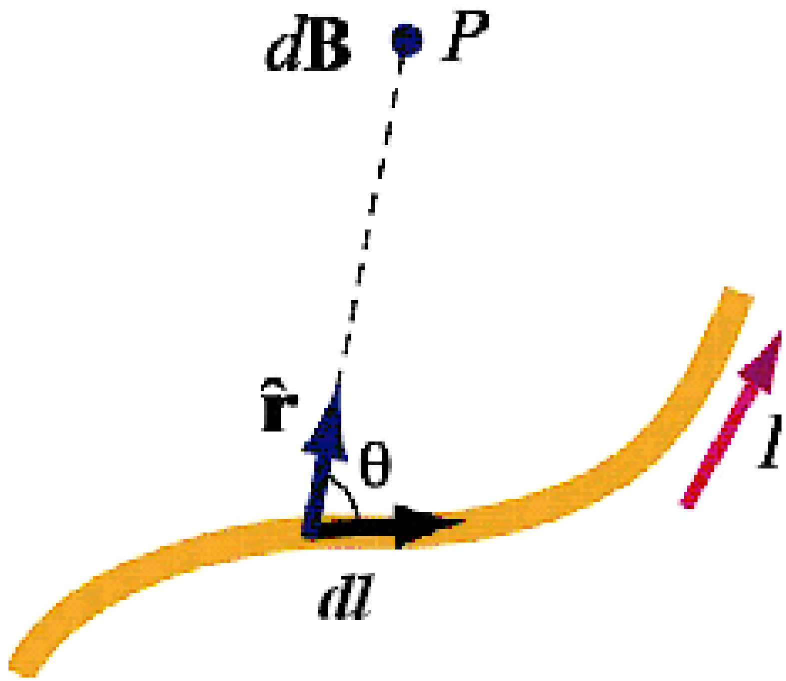
ii) Ampère's Law: An integral theorem similar to Gauss's law.

Biot-Savart Law

Current element of length dl carrying current (I) produces a magnetic field dB at point P:

$$dB \propto \frac{Idl \times \hat{r}}{r^2}$$

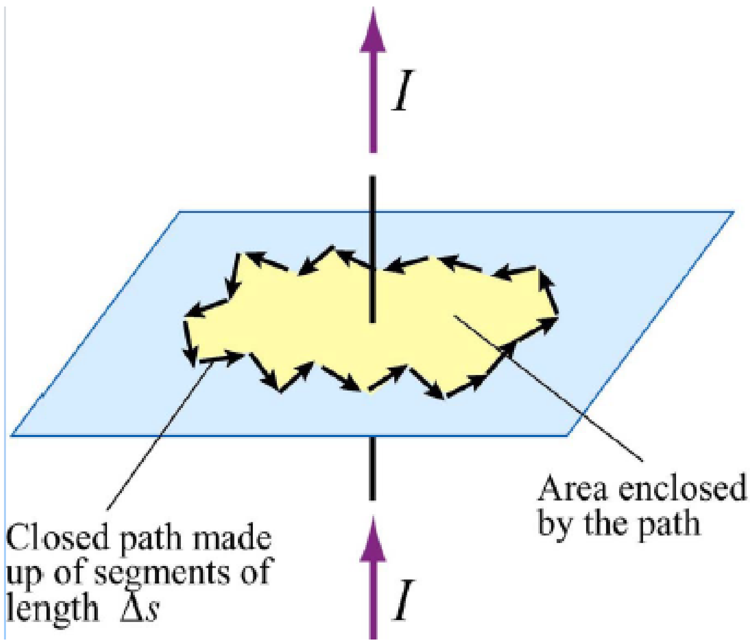
$$dB = \frac{\mu_0 Idl \sin\theta}{4\pi r^2}$$



μ_0– permeability of free space

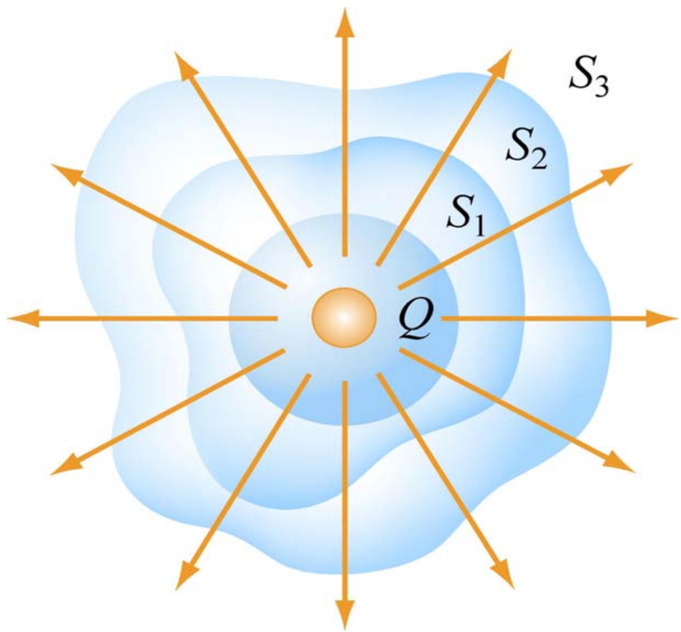
$\mu_0 = 4\pi \times 10^{-7}$ Wb/A.m

Ampere's law – The Idea



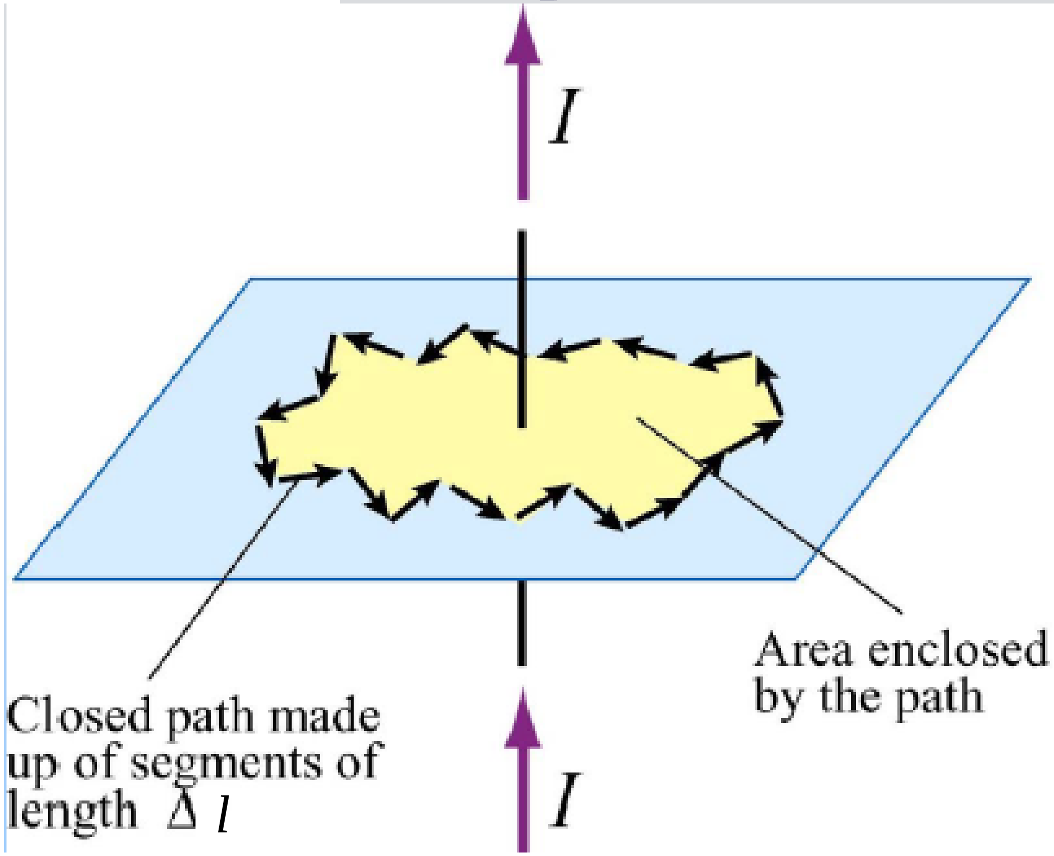
$$\oint \vec{\mathbf{B}} \cdot d\vec{l} = \mu_0 I_{enc}$$

Gauss's Law – The Idea



$$\Phi_E = \oint\!\!\!\oint_{\text{closed surfaceS}} \vec{\mathbf{E}} \cdot d\vec{\mathbf{A}} = \frac{q_{in}}{\varepsilon_0}$$

Ampere's Law: The Idea



In order to have a B field around a loop, there must be current punching through the loop

$$\oint \vec{\mathbf{B}} \cdot d\vec{l} = \mu_0 I_{enc}$$

Ampere's law

- **Ampere's law states that the line integral of B·*dl* around any closed loop is proportional to the total current passing through any surface that is bounded by the closed loop:**

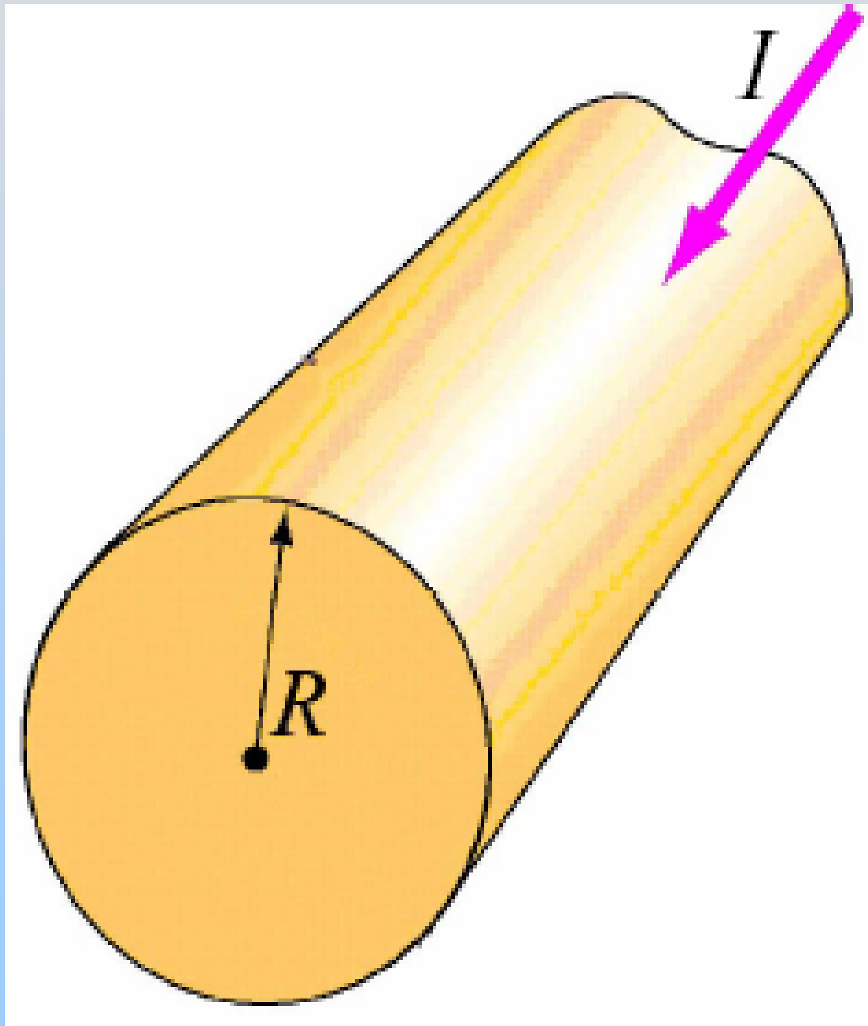
$$\oint_{c} B \bullet dl = \mu_0 I_{enc}$$

Applying Ampere's Law

1. Identify regions in which to calculate B field
2. Choose Amperian Loops S: Symmetry
3. Calculate $\oint \vec{\mathbf{B}} \cdot d\vec{l}$
4. Calculate current enclosed by loop S
5. Apply Ampere's Law to solve for B

$$\oint \vec{\mathbf{B}} \cdot d\vec{l} = \mu_0 I_{enc}$$

Example: Infinite Wire



A cylindrical conductor has radius R and a uniform current density with total current I

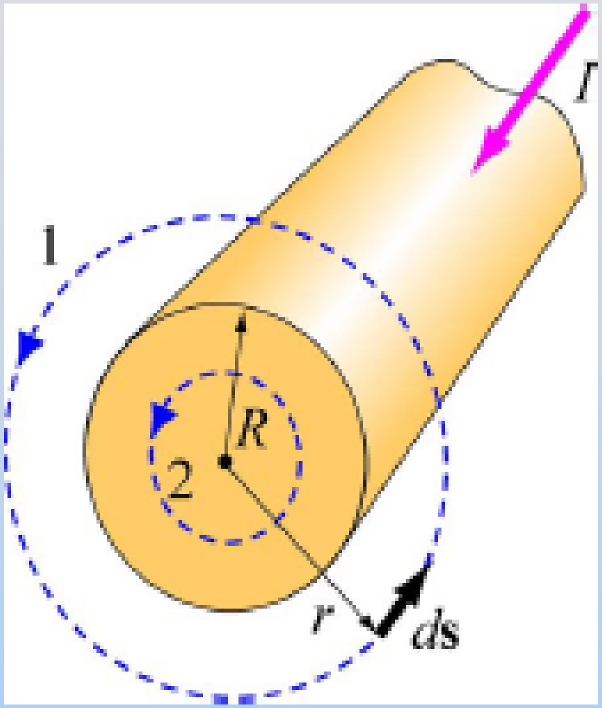
Find B everywhere

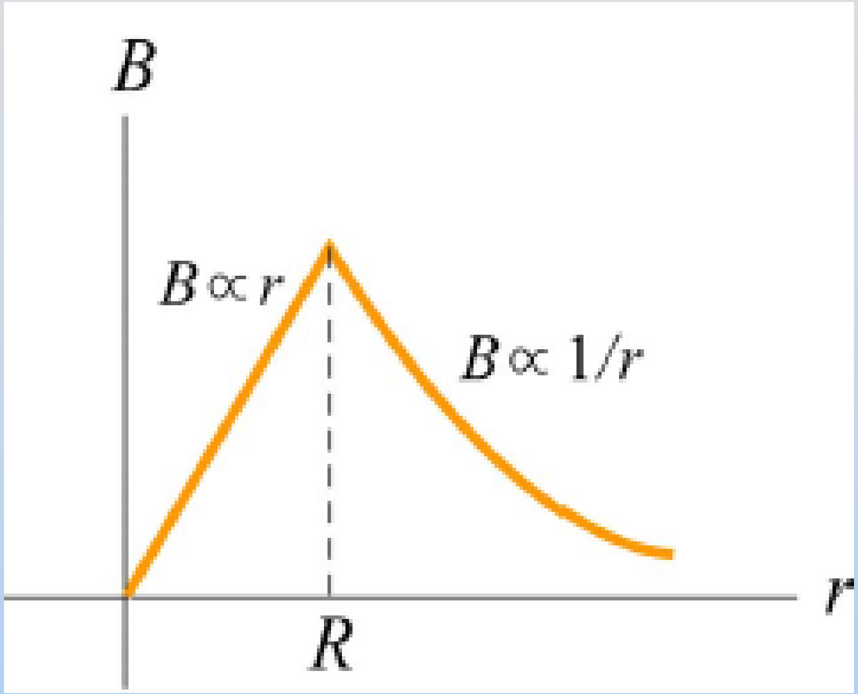
Two regions:

(1) outside wire ($r \geq R$)

(2) inside wire ($r < R$)

Example: Wire of Radius *R*





$$B_{in} = \frac{\mu_0 I r}{2\pi R^2} \qquad B_{out} = \frac{\mu_0 I}{2\pi r}$$

Maxwell's Equations

E = electric field

D = electric displacement

H = magnetic field

B = magnetic flux density

ρ= charge density

$\boldsymbol{j}_c$= current density

μ_0 (permeability of free space) = $4\pi\ 10^{-7}$

ε_0 (permittivity of free space) = $8.854\ 10^{-12}$

c (speed of light) = $2.99792458\ 10^{8}$ m/s

In vacuum

$$\left.\begin{aligned}\vec{D} &= \varepsilon_0 \vec{E} \\ \vec{B} &= \mu_0 \vec{H}\end{aligned}\right\} \quad \text{and} \quad \varepsilon_0 \mu_0 = \frac{1}{c^2}$$

so $\nabla \cdot \vec{E} = \rho / \varepsilon_0$

$$\nabla \cdot \vec{B} = 0$$

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

$$\nabla \times \vec{B} = \mu_0 \vec{j}_c + \frac{1}{c^2}\frac{\partial \vec{E}}{\partial t}$$

Maxwell's Equations in free space

Volume charge density ρ = 0 and Current density J = 0

$$\nabla \cdot \vec{D} = 0 \qquad .....(1)$$

$$\nabla \cdot \vec{B} = 0 \qquad .....(2)$$

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \qquad ....(3)$$

$$\nabla \times \vec{H} = \frac{\partial \vec{D}}{\partial t} \qquad ....(4)$$

Maxwell Equation 1.

$$\nabla \cdot E = \frac{\rho}{\epsilon_0}$$

Proof: According Gauss Law:

$$\oint E \cdot ds = \frac{q_{in}}{\epsilon_0}$$

where $q_{in} = \int_V \rho \, dV$

$$\oint E \cdot ds = \frac{\int_V \rho \, dV}{\epsilon_0} \quad \text{—— (1)}$$

Now according Gauss Divergence Theorem:

$$\oint E \cdot dl = \int_V (\nabla \cdot E) \cdot dV \quad \text{—— (2)}$$

Now from (1) & (2)

$$\int_V (\nabla \cdot E)\, dV = \int_V \frac{\rho}{\epsilon_0}\, dV$$

$$\Rightarrow \boxed{\nabla \cdot E = \frac{\rho}{\epsilon_0}}$$

Hence Proved

MAXWELL EQUATION 2

$$\nabla \cdot B = 0$$

Proof: According Bio Savart's law

$$dB = \frac{\mu_0 I \, dl \sin\theta}{4\pi r^2}$$

At $\theta = \pi/2$

$$dB = \frac{\mu_0 I \, dl}{4\pi r^2}$$

$$B = \frac{\mu_0}{4\pi} \oint \frac{I}{r^2} dl \qquad \text{——— (1)}$$

$$I = \oint J \cdot dS$$

$$B = \frac{\mu_0}{4\pi} \oint \frac{\oint J \cdot ds}{r^2} \, dl \quad \text{—— (2)}$$

Now Gauss Divergence Theorem.

$$\oint J \cdot dS = \int_V \nabla \cdot J \, dV$$

$$B = \frac{\mu_0}{4\pi} \oint \int_V \frac{\nabla \cdot J \, dV}{r^2} \, dl$$

Now taking Divergence Both side

$$\nabla \cdot B = \frac{\mu_0}{4\pi} \oint \int_V \frac{\nabla \cdot (\nabla \cdot J)}{r^2} \, dV \, dl$$

as $\nabla \cdot (\nabla \cdot J) = 0$

($\nabla \cdot J$ ⇓ scalar)

always $\Rightarrow \nabla \cdot (\text{scalar}) = 0$

$$\Rightarrow \boxed{\nabla \cdot B = 0}$$

Hence Proved.

Maxwell Equation 3

$$\nabla \times E = -\frac{dB}{dt}$$

Proof:

The electromotive force round a circuit

$$= \oint_C F \cdot dl$$

$$= \oint_C q E \cdot dl$$

$F = qE$

$q = 1$ Coulomb

$$\boxed{E.M.F. = \oint E \cdot dl} \quad \text{--- (1)}$$

Now "faraday law"

E.M.F. induced in a closed Loop is given by

$$E.M.F. = -\frac{d\phi}{dt}$$ ϕ = Magnetic flux

$$\phi = \int B \cdot dS$$

$$E.M.F. = -\frac{d}{dt}\int B \cdot dS \quad \text{--- (2)}$$

Now By Stokes theorem

$$\oint E \cdot dl = \int_S (\nabla \times E) \cdot ds \quad — ③$$

from ① & ③

$$E.M.F = \int_S (\nabla \times E) \cdot ds \quad — ④$$

Now from ② & ④.

$$\int_S (\nabla \times E) \cdot ds = \int_S -\frac{dB}{dt} \cdot ds$$

$$\nabla \times E = -\frac{dB}{dt}$$

MAXWELL EQUATION - 4

$$\nabla \times H = J_c + \frac{\partial D}{\partial t}$$

Proof: According Ampere circuital Law.

$$\oint H \cdot dl = I_{in}$$

$$I_{in} = \int_S J_c \cdot ds$$

$$\oint H \cdot dl = \int_S J_c \cdot ds \quad \text{--- (1)}$$

Now By Stokes Theorem:

$$\oint H \cdot dl = \int_S (\nabla \times H) \cdot ds \quad \text{--- (2)}$$

from (1) & (2)

$$\nabla \times H = J_c \quad \text{---} \quad (3)$$

Now take divergence of equation (3) both side

$$\nabla \cdot (\nabla \times H) = \nabla \cdot J_c \Rightarrow 0 \quad \text{always}$$

⇓

always (zero) ⇒

But the from the continuity equation:

$$\nabla \cdot J = -\frac{\partial \rho}{\partial t}$$

$$\nabla \cdot J = 0 \quad \text{only if} \quad \frac{\partial \rho}{\partial t} = 0$$

that mean current density is not varying with time but this is not always.

$$\frac{\partial \rho}{dt} \neq 0$$

so equation (3) does not provide the complete Proof

Maxwell assumed that the definition of current density J is incomplete and some thing say J_d must be added to it

then total current density

$$J = J_c + J_d$$

Now $\oint H \cdot dl = J_c + J_d$

$$\nabla \times H = J_c + J_d$$

Now taking divergence both side

$$\nabla \cdot (\nabla \times H) = \nabla \cdot (J_c + J_d)$$

$$\nabla \cdot (\nabla \times H) = 0$$

$$\nabla \cdot J_c + \nabla \cdot J_d = 0$$

$$\nabla \cdot J_d = -\nabla \cdot J_c$$

Now from continuity equation

$$\nabla \cdot J_c = -\frac{\partial \rho}{dt}$$

$$\nabla \cdot J_d = \frac{\partial \rho}{\partial t}$$

Now from Maxwell equation (1)

$$\nabla \cdot D = \rho$$

$$\nabla \cdot J_d = \frac{\partial}{\partial t} \nabla \cdot D$$

$$\nabla \cdot J_d = \nabla \cdot \frac{\partial D}{\partial t}$$

$$\Rightarrow \quad J_d = \frac{\partial D}{\partial t}$$

so equation (3) Modified to

$$\boxed{\nabla \times H = J_c + \frac{\partial D}{\partial t}}$$